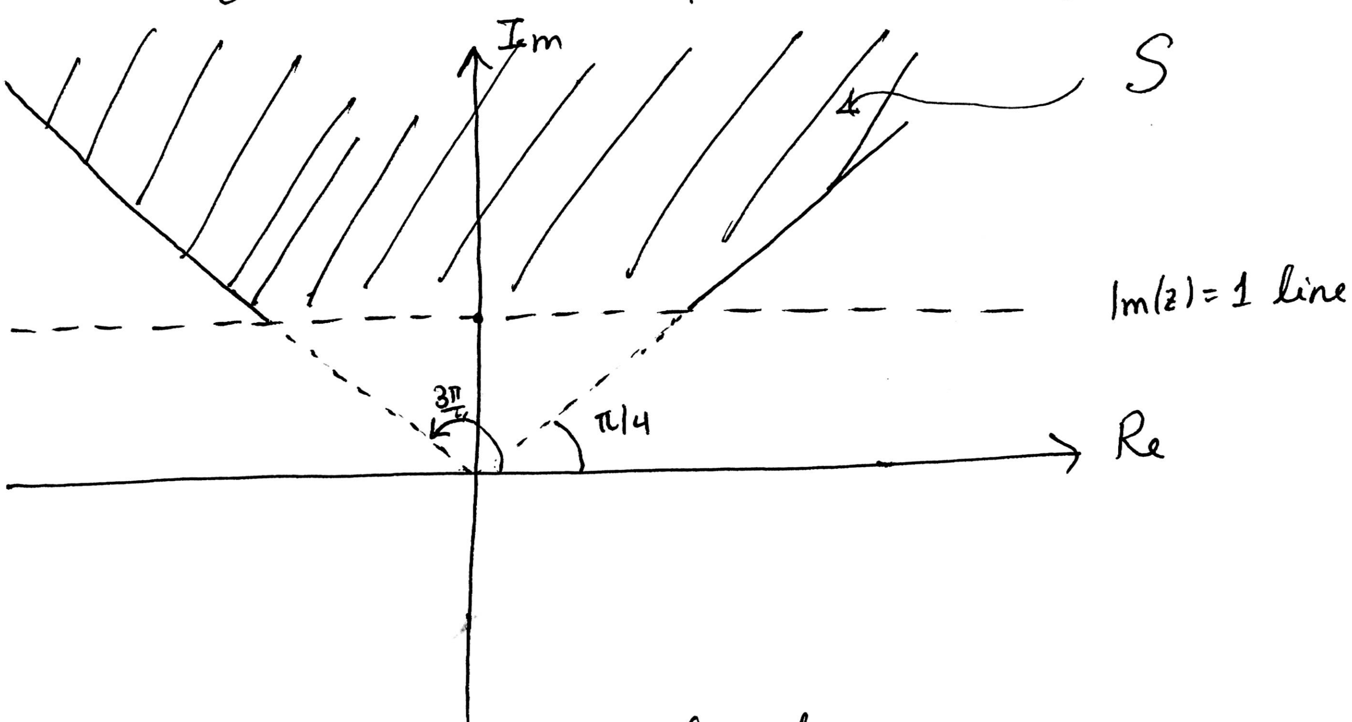
Practice Exam 1 - Solutions.

1. $S = \left\{ \mathrm{Im}(z) > 1 \text{ and } \frac{\pi}{4} \leq \arg(z) \leq \frac{3\pi}{4} \right\}$



- S is neither open nor closed.
- S is not bounded.
- S is connected.

2. (a) $\lim\limits_{z \to 0} \dfrac{e^{z^2} - 1}{z} = \lim\limits_{z \to 0} \dfrac{\frac{d}{dz}(e^{z^2} - 1)}{\frac{d}{dz}(z)}$ by l'hôpital rule

$= \lim\limits_{z \to 0} \dfrac{e^{z^2} \cdot 2z}{1} = e^0 \cdot 2 \cdot 0 = 0.$

(b) $\lim_{z \to 0} \frac{\sin(3z)}{2z} = \lim_{z \to 0} \frac{\frac{d}{dz}(\sin(3z))}{\frac{d}{dz}(2z)}$ (l'hôpital rule)

$= \lim_{z \to 0} \frac{\cos(3z) \cdot 3}{2} = \frac{3}{2}.$

(c) $\lim_{z \to 0} \frac{z^2 + z - i}{2z+3} = \frac{0^2 + 0 - i}{2(0)+3} = \frac{-i}{3}$

3. $u(x,y) = e^{-y}\left((x+1)\cos(x) - y\sin(x)\right)$

$\Rightarrow u_x = e^{-y}\left(\cos(x) - (x+1)\sin(x) - y\cos(x)\right)$

$u_y = -e^{-y}\left((x+1)\cos(x) - y\sin(x)\right) - e^{-y}\sin(x)$

$v(x,y) = e^{-y}\left((x+1)\sin(x) + y\cos(x)\right)$

$\Rightarrow v_x = e^{-y}\left(\sin(x) + (x+1)\cos(x) - y\sin(x)\right)$

$v_y = -e^{-y}\left((x+1)\sin(x) + y\cos(x)\right) + e^{-y}\cos(x)$

C-R eq^{n}s: $u_x = v_y$ ✓

$u_y = -v_x$ ✓

$$f'(z) = u_x + i v_x$$

$$= e^{-y}\left(\cos(x) - (x+1)\sin(x) - y\cos(x)\right)$$

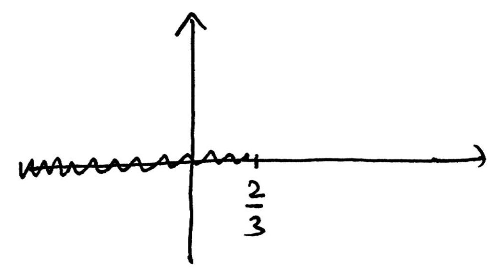
$$+ i e^{-y}\left(\sin(x) + (x+1)\cos(x) - y\sin(x)\right)$$

4. (a) $\ln(3z-2)$ is defined for all $z \in \mathbb{C}$ such that

$$3z - 2 \notin \mathbb{R}_{\leq 0}$$

i.e. $3z \notin \mathbb{R}_{\leq 2}$ meaning $z \notin \mathbb{R}_{\leq \frac{2}{3}}$

Domain $= \mathbb{C} \setminus \mathbb{R}_{\leq \frac{2}{3}}$



$$\frac{d}{dz}\ln(3z-2) = \frac{3}{3z-2}.$$

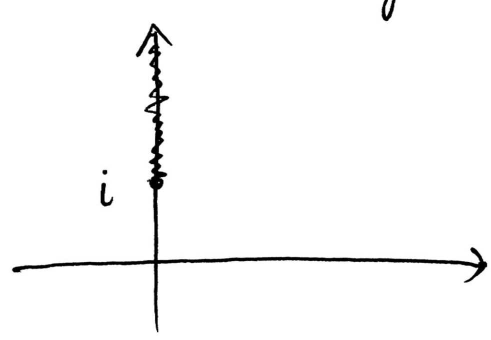
(b) $(iz+1)^{\frac{1}{3}} = e^{\frac{1}{3}\ln(iz+1)}$ is defined for z

such that $iz + 1 \notin \mathbb{R}_{\leq 0}$

$iz \notin \mathbb{R}_{\leq -1}$; i.e. $z \neq t \cdot i$ for any $t \in \mathbb{R}_{\geq 1}$.

(Note: $iz = -t$ means $z = it$)

Domain $=$



$$\frac{d}{dz}(iz+1)^{\frac{1}{3}} = \frac{1}{3}(iz+1)^{-2/3} \cdot i$$

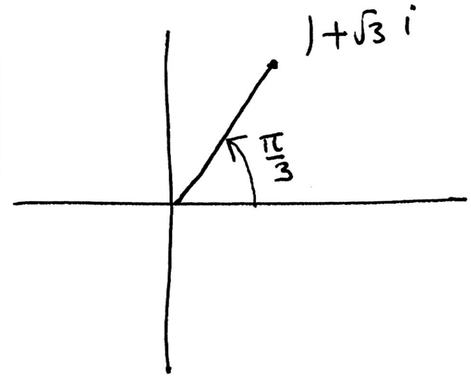
5. $$(1+\sqrt{3}\,i)^{3\pi i} = e^{3\pi i \ln(1+\sqrt{3}\,i)}$$

$$= e^{3\pi i\left(\ln(2) + i\cdot\frac{\pi}{3}\right)}$$

$$= e^{-\pi^2} \cdot e^{i\cdot(3\pi \ln(2))}$$

Real part $= e^{-\pi^2}\cos(3\pi\cdot\ln(2))$

Imaginary part $= e^{-\pi^2}\sin(3\pi\ln(2))$



$$|1+\sqrt{3}\,i| = 2$$

$$\arg(1+\sqrt{3}\,i) = \frac{\pi}{3}$$

6. $$\sin(z) = \frac{e^{iz} - e^{-iz}}{2i} \qquad \cos(z) = \frac{e^{iz}+e^{-iz}}{2}$$

$$\sin(2z)\sin(z) = \frac{\left(e^{2iz} - e^{-2iz}\right)\left(e^{iz} - e^{-iz}\right)}{(2i)(2i)}$$

$$= \frac{e^{3iz} - e^{iz} - e^{-iz} + e^{-3iz}}{-4}$$

$$= \frac{e^{iz} + e^{-iz} - \left(e^{3iz} + e^{-3iz}\right)}{4}$$

(Use $A^3 + B^3 = (A+B)\cdot(A^2 - AB + B^2)$)

$$= \frac{\left(e^{iz}+e^{-iz}\right)\left(1-\left(e^{2iz}-1+e^{-2iz}\right)\right)}{4}$$

$$= \frac{e^{iz}+e^{-iz}}{2} \cdot \frac{2-\left(e^{2iz}+e^{-2iz}\right)}{2}$$

$$= \cos(z) \cdot \left(1-\cos(2z)\right)$$

7. $z^6+1=0 \Rightarrow z^6 = -1 = e^{\pi i}$

$$\Rightarrow z = e^{\frac{\pi}{6}i},\ e^{\left(\frac{\pi}{6}+\frac{2\pi}{6}\right)i},\ \ldots,\ e^{\left(\frac{\pi}{6}+5\cdot\frac{2\pi}{6}\right)i}$$

$$= e^{\frac{\pi}{6}i},\ e^{\frac{3\pi}{6}i},\ e^{\frac{5\pi}{6}i},\ \ldots,\ e^{\frac{11\pi}{6}i}$$

Product of these six numbers

$$= e^{\frac{\pi}{6}i\,(1+3+5+7+9+11)} = e^{\frac{\pi}{6}i\cdot 36}$$

$$= e^{6\pi i} = 1.$$