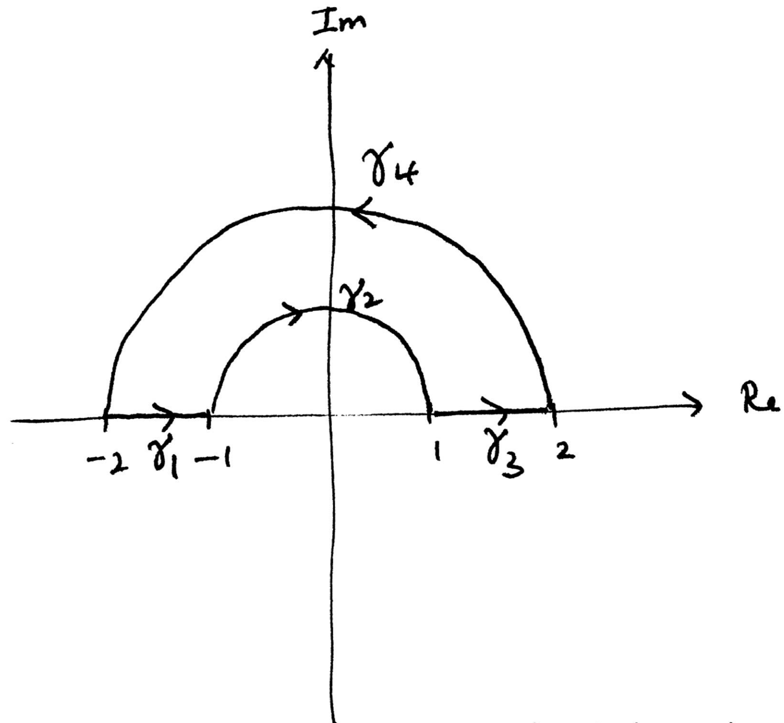
Solutions to Practice Exam 2

(1)



[Contour C - consists of 4 parts $\gamma_1, \gamma_2, \gamma_3, \gamma_4$]

- $\gamma_1 : [-2,-1] \to \mathbb{C}$
  $\gamma_1(t) = t$
- $\gamma_2 : [0,\pi] \to \mathbb{C}$
  $\gamma_2(t) = -e^{-it}$
- $\gamma_3 : [1,2] \to \mathbb{C}$
  $\gamma_3(t) = t$
- $\gamma_4 : [0,\pi] \to \mathbb{C}$
  $\gamma_4(t) = 2 \cdot e^{it}$.

(i) $$\int_{\gamma_1} \frac{z}{\bar{z}}\, dz = \int_{-2}^{-1} \frac{t}{t} \cdot 1 \cdot dt = [t]_{-2}^{-1} = -1-(-2) = 1.$$

(ii) $$\int_{\gamma_3} \frac{z}{\bar{z}}\, dz = \int_{1}^{2} \frac{t}{t} \cdot 1 \cdot dt = [t]_1^2 = 2-1 = 1.$$

(iii) $$\int_{\gamma_2} \frac{z}{\bar{z}}\, dz = \int_0^{\pi} \frac{\cancel{-}e^{-it}}{\cancel{-}e^{+it}} \cdot (\cancel{+1})(\cancel{+}i)e^{-it}\, dt = i \int_0^{\pi} e^{-3it}\, dt$$

$$= i \left[\frac{e^{-3it}}{-3i} \right]_0^{\pi} = -\frac{1}{3}\left(e^{-3\pi i} - e^{0}\right) = \frac{-1}{3}(-1-1) = \frac{2}{3}.$$

(iv) $$\int_{\gamma_4} \frac{z}{\bar{z}}\,dz = \int_0^{\pi} \frac{\cancel{2}\cdot e^{it}}{\cancel{2}\cdot e^{-it}}\cdot 2\cdot i\cdot e^{it}\,dt$$

$$= 2i\int_0^{\pi} e^{3it}\,dt = 2i\left[\frac{e^{3it}}{3i}\right]_0^{\pi} = \frac{2}{3}(-1-1) = \frac{-4}{3}.$$

$$\Rightarrow \int_C \frac{z}{\bar{z}}\,dz = 1+1+\frac{2}{3}-\frac{4}{3} = 2-\frac{2}{3} = \frac{4}{3}.$$

---

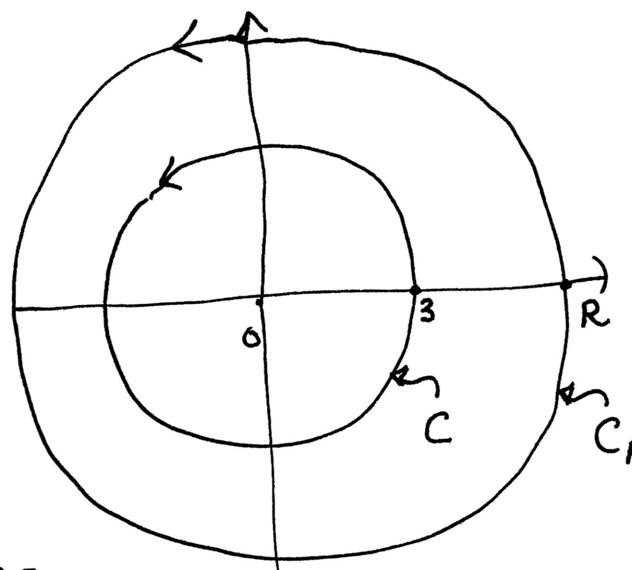
2 $\Omega = \{z \mid |z|>2\}$. $f:\Omega\to\mathbb{C}$ holomorphic.

s.t. $|f(z)| < \dfrac{1}{|z|^2}$.

Principle of Contour deformation:

$$\int_C f(z)\,dz = \int_{C_R} f(z)\,dz$$

(for every $R>3$).



Important inequality

$$\left|\int_{C_R} f(z)\,dz\right| \leq \frac{1}{R^2}\cdot 2\pi R = \frac{2\pi}{R}.$$

Hence $\left|\int_C f(z)\,dz\right| \leq \frac{2\pi}{R}$ for every $R>3$.

$$\Rightarrow \left| \int_C f(z)\,dz \right| = 0 \Rightarrow \int_C f(z)\,dz = 0.$$

---

3 (a) Antiderivative of $\sin^3(z) = \sin^2(z)\cdot\sin(z)$

$$= (1-\cos^2(z))\cdot\sin(z)$$

$$u = \cos(z) \rightsquigarrow \int (1-u^2)(-du) = \frac{u^3}{3} - u.$$

So, antiderivative of $\sin^3(z) = \dfrac{\cos^3(z)}{3} - \cos(z)$

$$\left(\text{Check: } \frac{d}{dz}\left(\frac{\cos^3(z)}{3} - \cos(z)\right) = \frac{\cancel{3}\cos^2(z)(-\sin(z))}{\cancel{3}} + \sin(z)\right.$$

$$\left. = \sin(z)(1-\cos^2(z)) = \sin^3(z) \checkmark \right)$$

$$\Rightarrow \int_\gamma \sin^3(z)\,dz = \left[\frac{1}{3}\cos^3(z) - \cos(z)\right]_{z=0}^{z=1+i}$$

(γ path from 0 to $1+i$)

$$= \left(\frac{1}{3}\cos^3(1+i) - \cos(1+i)\right) - \left(\frac{1}{3} - 1\right)$$

$$= \frac{1}{3}\cos^3(1+i) - \cos(1+i) + \frac{2}{3}.$$

(b) $f(z) = (e^z - 1)^2 = e^{2z} - 2\cdot e^z + 1$

Antiderivative : $F(z) = \dfrac{e^{2z}}{2} - 2\cdot e^z + z$

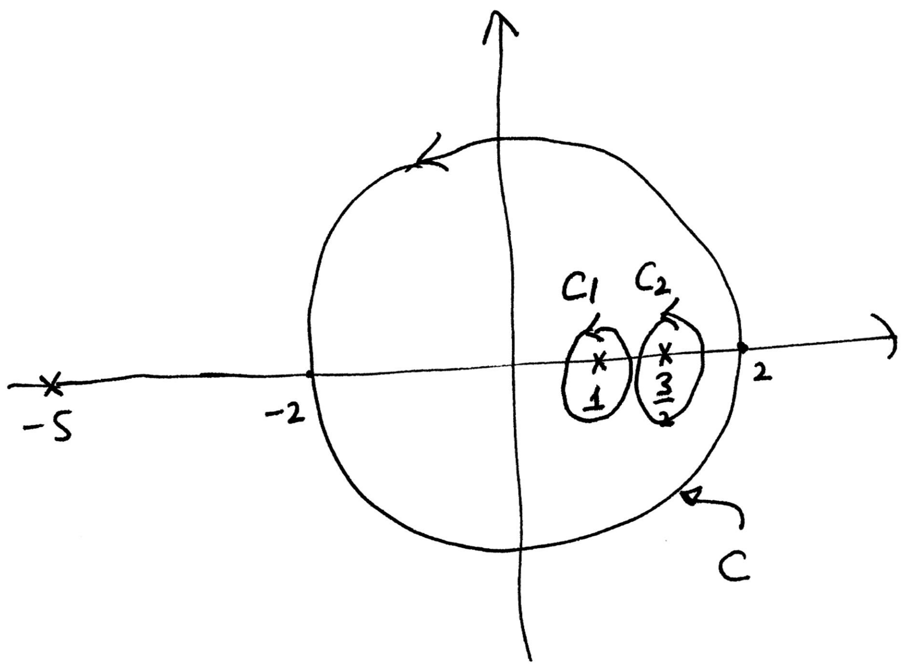
$$\int_\gamma f(z)\,dz = F(1+i) - F(0)$$

$$= \left(\frac{e^{2(1+i)}}{2} - 2e^{1+i} + (1+i)\right) - \left(\frac{1}{2} - 2 + 0\right)$$

$$= \frac{1}{2}e^{2(1+i)} - 2e^{1+i} + 1 + i + \frac{1}{2}.$$

4

$$\int_C \frac{z^2-2}{(z+5)(2z-3)(z-1)}\,dz$$ (by principle of contour deformation)



$$= \int_{C_1} \frac{1}{z-1}\cdot\frac{z^2-2}{(z+5)(2z-3)}\,dz$$

$$+ \int_{C_2} \frac{1}{z-\frac{3}{2}}\cdot\frac{1}{2}\cdot\frac{z^2-2}{(z+5)(z-1)}\,dz$$

$$= 2\pi i\left[\frac{z^2-2}{(z+5)(2z-3)}\right]_{\text{set } z=1} + 2\pi i\left[\frac{1}{2}\,\frac{z^2-2}{(z+5)(z-1)}\right]_{\text{set } z=\frac{3}{2}}$$

Cauchy's integral formula

$$= 2\pi i\left(\frac{-1}{6\cdot(-1)}\right) + 2\pi i\left(\frac{\frac{9}{4}-2}{\cancel{2}\left(\frac{3}{2}+5\right)\left(\frac{1}{\cancel{2}}\right)}\right) \qquad \left[\frac{\frac{1}{4}}{\frac{13}{2}} = \frac{1}{26}\right]$$

$$= 2\pi i\left(\frac{1}{6}+\frac{1}{26}\right) = 2\pi i\left(\frac{13+3}{78}\right) = 2\pi i\left(\frac{16}{78}\right).$$

5. $$\frac{2z-1}{(z-3)(z-i)^2} = \frac{A}{z-3} + \frac{B}{z-i} + \frac{C}{(z-i)^2}$$

$$\Rightarrow 2z-1 = A(z-i)^2 + B(z-3)(z-i) + C(z-3) \quad - (*)$$

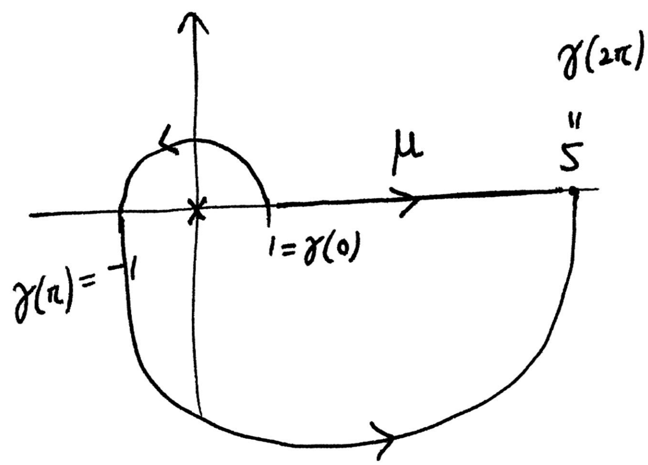
Set $z = 3$ in $(*)$: $\quad 5 = A\cdot(3-i)^2 \Rightarrow \boxed{A = \frac{5}{(3-i)^2}}$.

Set $z = i$ in $(*)$: $\quad 2i-1 = C(i-3) \Rightarrow \boxed{C = \frac{1-2i}{3-i}}$

Compare coeff. of z^2 in $(*)$: $\quad 0 = A+B \Rightarrow \boxed{B = -A = \frac{-5}{(3-i)^2}}$.

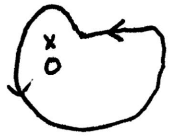
6. $\gamma : [0, 2\pi] \to \mathbb{C}$

$$\gamma(t) = \begin{cases} e^{it} ; & 0 \leq t \leq \pi \\ 2+3e^{it} ; & \pi \leq t \leq 2\pi \end{cases}$$



Let μ be the straight line from 1 to 5.

$$\int_\gamma \frac{1}{z}\,dz - \int_\mu \frac{1}{z}\,dz = \int \frac{1}{z}\,dz = 2\pi i \quad \text{(Cauchy's formula)}$$



$$\Rightarrow \int_\gamma \frac{1}{z}\,dz = 2\pi i + \int_1^5 \frac{1}{t}\,dt = 2\pi i + \ln(5).$$