COMPLEX ANALYSIS: PRACTICE EXAM 2

Problem 1. Let C be the contour, consisting of the following four pieces:

- From -2 to -1 along a straight line.
- From -1 to $+1$ along the semicircular arc (radius 1, centered at 0) lying above the real axis.
- From 1 to 2 along a straight line.
- From 2 to -2 along the semicircular arc (radius 2, centered at 0) lying above the real axis.

(a) Write each of the four smooth parts of C in parametric form.

(b) Compute $\displaystyle\int_C \frac{z}{\bar{z}}\,dz$.

Problem 2. Let $f(z)$ be a holomorphic function, defined on $\Omega = \{z \in \mathbb{C} : |z| > 2\}$. Assume that $|f(z)| < \dfrac{1}{|z|^2}$, for every $z \in \Omega$. Let C be the counterclockwise circle of radius 3, centered at 0. Prove that $\displaystyle\int_C f(z)\,dz = 0$.

Problem 3. Let γ be any piecewise smooth path starting at 0 and ending at $1 + \mathbf{i}$. Compute $\displaystyle\int_\gamma f(z)\,dz$ for each of the following functions.

(a) $f(z) = \sin^3(z)$, (b) $f(z) = (e^z - 1)^2$.

Problem 4. Let C be the counterclockwise circle of radius 2, centered at 0. Compute

$$\int_C \frac{z^2 - 2}{(z+5)(2z-3)(z-1)}\,dz.$$

Problem 5. Compute the partial fraction decomposition of $\dfrac{2z - 1}{(z-3)(z-\mathbf{i})^2}$.

Problem 6. Let $\gamma : [0, 2\pi] \to \mathbb{C}$ be given by:

$$\gamma(t) = \begin{cases} e^{\mathbf{i}t}, & 0 \le t \le \pi, \\ 2 + 3e^{\mathbf{i}t}, & \pi \le t \le 2\pi. \end{cases}$$

Prove that $\displaystyle\int_\gamma \frac{1}{z}\,dz = \ln(5) + 2\pi\mathbf{i}$.