# Review I

(R1.0) Numbers. - $\mathbb{Z}$ = set of integers $= \{0, \pm 1, \pm 2, \dots\}$.

$\mathbb{R}$ = set of real numbers.

$\mathbb{C}$ = set of complex numbers.

$z \in \mathbb{C}$ is written as:

$$z = x + yi = r(\cos(\theta) + \sin(\theta)i) = r \cdot e^{i\theta}$$

(Cartesian form) (polar form, $z \neq 0$.)

$x = \mathrm{Re}(z)$

$y = \mathrm{Im}(z)$

(real and imaginary components of $z$.)

$r = |z| = \sqrt{x^2 + y^2}$ (modulus of $z$)

$$\boxed{\cos(\theta) = \frac{x}{\sqrt{x^2+y^2}} \qquad \sin(\theta) = \frac{y}{\sqrt{x^2+y^2}}} \leftarrow \theta = \arg(z) \quad (-\pi < \theta \leq \pi)$$

Euler's formula: $\boxed{e^{i\theta} = \cos(\theta) + \sin(\theta)i}$ $(\theta \in \mathbb{R})$.

Conjugate of $z$: $\boxed{\begin{aligned} \bar{z} &= x - yi = r \cdot e^{-i\theta} \\ z \cdot \bar{z} &= |z|^2 \end{aligned}}$

Addition.

$$(x_1 + y_1 i) + (x_2 + y_2 i) = (x_1 + x_2) + (y_1 + y_2) i$$

Multiplication. $\boxed{i^2 = -1}$

$$(r_1 e^{i\theta_1})(r_2 e^{i\theta_2}) = r_1 r_2 e^{i(\theta_1 + \theta_2)}.$$

(R1.1) Let $S \subseteq \mathbb{C}$ be a subset of $\mathbb{C}$.

- We say $S$ is <u>open</u> if for every $\alpha \in S$, there is some $r \in \mathbb{R}_{>0}$ such that $D(\alpha; r) = \{z \mid |z-\alpha| < r\} \subset S$.

- We say $S$ is <u>closed</u> if its <u>complement</u> is open.
  $(\mathbb{C} \setminus S = \{z \mid z \notin S\})$

- We say $S$ is <u>bounded</u> if we can find $R > 0$ so that
  $\boxed{\alpha \in S \Rightarrow |\alpha| < R}$ (i.e, $S \subset D(0; R)$)

- We say $S$ is <u>connected</u> if any two $\alpha, \beta \in S$ can be joined by a path that lies entirely in $S$.

Remark. Often $S$ is given by a strict inequality $<$ ; an inequality $\leqslant$ ; or an equality $=$.
"$<$" $\leftrightarrow$ open ; "$\leqslant$" or "$=$" $\leftrightarrow$ closed (if both are involved the subset is neither open, nor closed.)

(R1.2) Let $f : \Omega \to \mathbb{C}$ be a complex-valued function, defined on an open set $\Omega \subseteq \mathbb{C}$.

- Cauchy - Riemann equations: If $f(z = x + yi) = u(x,y) + v(x,y)i$ then C-R eq$^{n}$s are: $\boxed{\dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y} \text{ and } \dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x}}$

- We say that $f$ is <u>$\mathbb{C}$-differentiable</u> at $z_0 \in \Omega$ if $\boxed{\lim_{\substack{h \to 0 \\ (h \in \mathbb{C})}} \frac{f(z_0+h)-f(z_0)}{h}}$ exists. <u>$f$ is $\mathbb{C}$-differentiable</u> means that it is $\mathbb{C}$-differentiable at every point $z_0 \in \Omega$.

- Once Cauchy-Riemann equations are verified, we get

$$\boxed{f'(z) = u_x + v_x \cdot i = v_y - u_y i}.$$

- A real-valued function of 2 real variables, say $H(x,y)$, is called <u>harmonic</u>, if it satisfies:

$$\boxed{H_{xx} + H_{yy} = 0} \quad \text{(Laplace equation).}$$

We proved that:

$$\left\{ \begin{array}{l} f = u(x,y) + v(x,y)i \\ \text{is } \mathbb{C}\text{-differentiable} \end{array} \right\} \Rightarrow \left\{ \begin{array}{c} u(x,y) \text{ and } v(x,y) \\ \text{are harmonic} \end{array} \right\}$$

Conversely, if $u(x,y)$ is harmonic, we can find $v(x,y)$ from C-R eq$^n$s, which makes $u(x,y)+v(x,y)i$; a $\mathbb{C}$-differentiable function:

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

$$\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x}$$

given. Solve for $v$.

(R 1.3) Examples of $\mathbb{C}$-differentiabe functions.

(i) $z^n$ $(n \in \mathbb{Z}_{\geq 0})$. Domain $= \mathbb{C}$. $\frac{d}{dz} z^n = n z^{n-1}$.

(more generally, polynomial functions:

$a_0 + a_1 z + \dots + a_n z^n$ ; where $a_0, \dots, a_n \in \mathbb{C}$, $n \in \mathbb{Z}_{\geq 0}$.)

(ii) $z^{-n}$ $(n \in \mathbb{Z}_{\geq 1})$. Domain $= \mathbb{C}^\times = \mathbb{C} \setminus \{0\}$.

More generally, rational functions $\frac{P(z)}{Q(z)}$, where P & Q are polynomials; Domain $= \mathbb{C} \setminus \{z \mid Q(z) = 0\}$.

(iii) $\boxed{e^z = e^x(\cos(y) + \sin(y)i)}$ Domain $= \mathbb{C}$.

$$\frac{d}{dz} e^z = e^z \; ; \quad e^0 = 1. \; (= e^{2n\pi i} \; ; \; n \in \mathbb{Z}.)$$

(iv) $$\boxed{\begin{aligned} \sin(z) &= \frac{e^{iz} - e^{-iz}}{2i} \\ \cos(z) &= \frac{e^{iz} + e^{-iz}}{2} \end{aligned}}$$ Domain $= \mathbb{C}$.

$$\frac{d}{dz} \sin(z) = \cos(z) \quad ; \quad \sin(0) = 0$$

$$\frac{d}{dz} \cos(z) = -\sin(z) \qquad \cos(0) = 1$$

(v) $$\boxed{\ln(z) = \ln(|z|) + \arg(z)i}$$

$$\text{Domain} = \{z \mid -\pi < \arg(z) < \pi\} = \mathbb{C} - \mathbb{R}_{\leq 0}.$$

$$\frac{d}{dz}\ln(z) = \frac{1}{z}. \qquad \ln(1) = 0.$$

(vi) $$\boxed{z^{\alpha} = e^{\alpha \ln(z)}} \qquad \alpha \in \mathbb{C},\ \alpha \notin \mathbb{Z}.$$

$$\text{Domain} = \mathbb{C} - \mathbb{R}_{\leq 0}.$$

$$\frac{d}{dz} z^{\alpha} = \alpha z^{\alpha - 1}; \qquad 1^{\alpha} = 1.$$

(R 1.4) Rules of differentiation:

- $(f+g)' = f' + g'$. • $(fg)' = f'g + fg'$.
- $\left(\frac{f}{g}\right)' = \frac{gf' - fg'}{g^2}$. • $\frac{d}{dz} f(g(z)) = f'(g(z))\, g'(z)$

L'Hôpital rule: Assume: $f(z_0) = g(z_0) = 0$. $f'(z_0)$, $g'(z_0)$ exist, and $g'(z_0) \neq 0$.

$$\boxed{\lim_{z \to z_0} \frac{f(z)}{g(z)} = \frac{f'(z_0)}{g'(z_0)}}$$

(R 1.5) $n^{th}$ roots. Solutions to $z^n = 1$ are:

$$\left\{1, e^{\frac{2\pi i}{n}}, e^{\frac{4\pi i}{n}}, \ldots, e^{\frac{(n-1)2\pi i}{n}}\right\}$$

If $\alpha = r \cdot e^{i\theta}$ $\quad (-\pi < \theta \leq \pi$, say);

solutions to $z^n = \alpha$ are: $\quad z_0 = r^{1/n} \cdot e^{\frac{i\theta}{n}}$ and

$$z_1 = z_0 \cdot e^{\frac{2\pi i}{n}}; \; z_2 = z_0 \cdot e^{\frac{4\pi i}{n}}; \; \ldots; \; z_{n-1} = z_0 \cdot e^{\frac{(n-1)2\pi i}{n}}.$$

(R 1.6) Useful algebraic identities to deal with $n^{th}$ roots:

$$\boxed{A^n - B^n = (A-B)(A^{n-1} + A^{n-2} \cdot B + A^{n-3} \cdot B^2 + \ldots + B^{n-1})}$$

Some values of sine and cosine

| $\theta$ | $0$ | $\frac{\pi}{6}$ | $\frac{\pi}{4}$ | $\frac{\pi}{3}$ | $\frac{\pi}{2}$ |
|---|---|---|---|---|---|
| $\sin(\theta)$ | $0$ | $\frac{1}{2}$ | $\frac{1}{\sqrt{2}}$ | $\frac{\sqrt{3}}{2}$ | $1$ |
| $\cos(\theta)$ | $1$ | $\frac{\sqrt{3}}{2}$ | $\frac{1}{\sqrt{2}}$ | $\frac{1}{2}$ | $0$ |

Binomial formula:

$$(A+B)^n = A^n + nA^{n-1}B + \frac{n!}{(n-2)!\,2!}A^{n-2}B^2 + \ldots$$

$$+ \frac{n!}{(n-k)!\,k!}A^{n-k}B^k + \ldots + B^n$$

denoted by $\binom{n}{k}$.