Review II

(R2.0) The focus of the second exam will be on $\int_\gamma f(z)\,dz$.

$f : \Omega \to \mathbb{C}$ is a continuous function, defined on an open set Ω.

$\gamma : [a,b] \to \Omega$ is a piecewise smooth path.

(R2.1) Definition and basic properties.

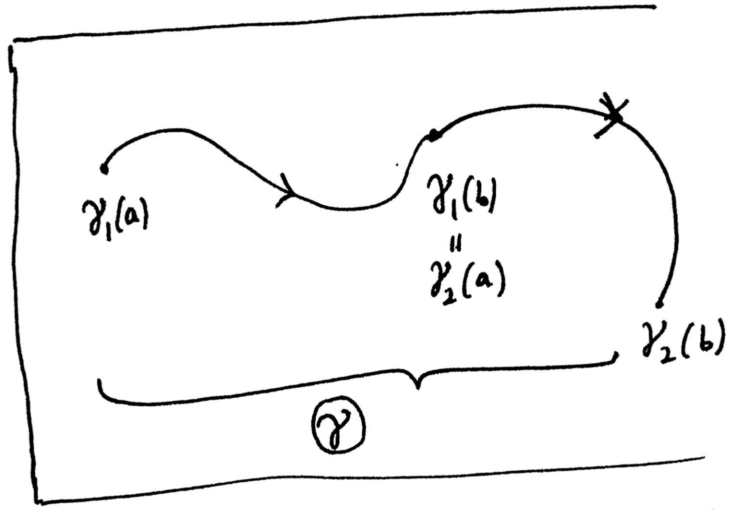
- If γ is smooth: $$\int_\gamma f(z)\,dz = \int_a^b f(\gamma(t))\,\gamma'(t)\,dt$$

- If γ is piecewise smooth: $\begin{cases} [a,b] = [a,c_1] \cup [c_1,c_2] \cup \ldots \cup [c_\ell, b] \\ \text{s.t. } \gamma \text{ is smooth on each subinterval} \\ [a,c_1],\ [c_1,c_2],\ \ldots,\ [c_\ell,b] \end{cases}$

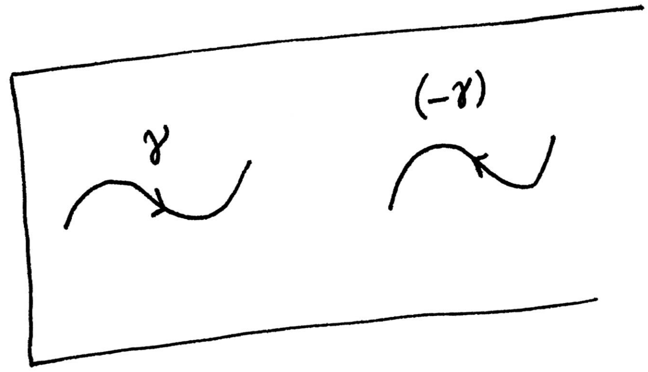
Then $$\int_\gamma f(z)\,dz = \int_a^{c_1} f(\gamma(t))\,\gamma'(t)\,dt + \int_{c_1}^{c_2} f(\gamma(t))\,\gamma'(t)\,dt + \cdots + \int_{c_\ell}^{b} f(\gamma(t))\,\gamma'(t)\,dt$$

Property 1. $\gamma = \gamma_1$ concatenated with γ_2, then

$$\int_{\gamma} f(z)\,dz = \int_{\gamma_1} f(z)\,dz + \int_{\gamma_2} f(z)\,dz$$



Property 2.
$$\int_{(-\gamma)} f(z)\,dz = -\int_{\gamma} f(z)\,dz$$



Property 3. (Important inequality).

If $M \in \mathbb{R}_{\geq 0}$ is such that $|f(\gamma(t))| \leq M$ for every $t \in [a,b]$,

and $L = \text{length}(\gamma)$, then

$$\left| \int_{\gamma} f(z)\,dz \right| \leq M \cdot L$$

(R2.2) • If $f: \Omega \to \mathbb{C}$ is <u>NOT</u> holomorphic, the <u>only way</u> to compute $\int_\gamma f(z)\,dz$ is from <u>the definition</u>.

• Some useful parametrizations:

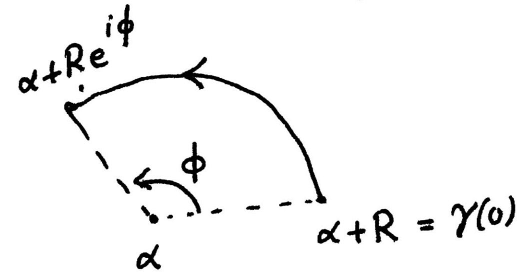
(i) γ = straight line joining $\alpha \in \mathbb{C}$ to $\beta \in \mathbb{C}$:

$$\gamma : [0,1] \to \mathbb{C} \;;\; \boxed{\gamma(t) = \alpha + t(\beta - \alpha)}.$$

(ii) γ = counterclockwise circular arc, centered at α, of radius R, starting at $\alpha + R$ (angle 0), ending at $\alpha + Re^{i\phi}$ (angle ϕ):

$$\gamma : [0, \phi] \to \mathbb{C}$$

$$\underline{\gamma(t) = \alpha + R \cdot e^{it}}.$$



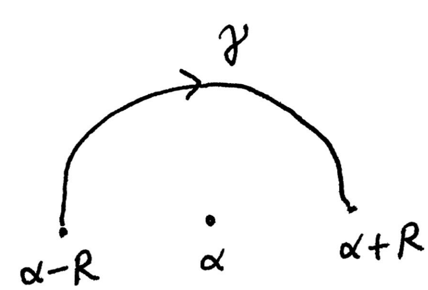
[Remember: $e^{i\pi} = -1$, $e^{i\frac{\pi}{2}} = i$

$e^{-i\pi} = -1$, $e^{-i\frac{\pi}{2}} = -i$.]

(iii) γ = clockwise arc, starting at $\alpha - R$, ending at $\alpha + R$:

$$\gamma : [0, \pi] \to \mathbb{C} :$$

$$\underline{\gamma(t) = \alpha - Re^{-it}}.$$

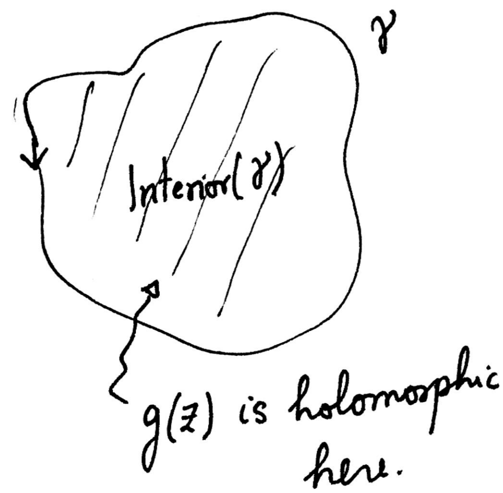


(R2.3) When g ~~f~~ $: \Omega \to \mathbb{C}$ is holomorphic, we have very useful tools to compute $\int_\gamma g(z)\,dz$. – namely:

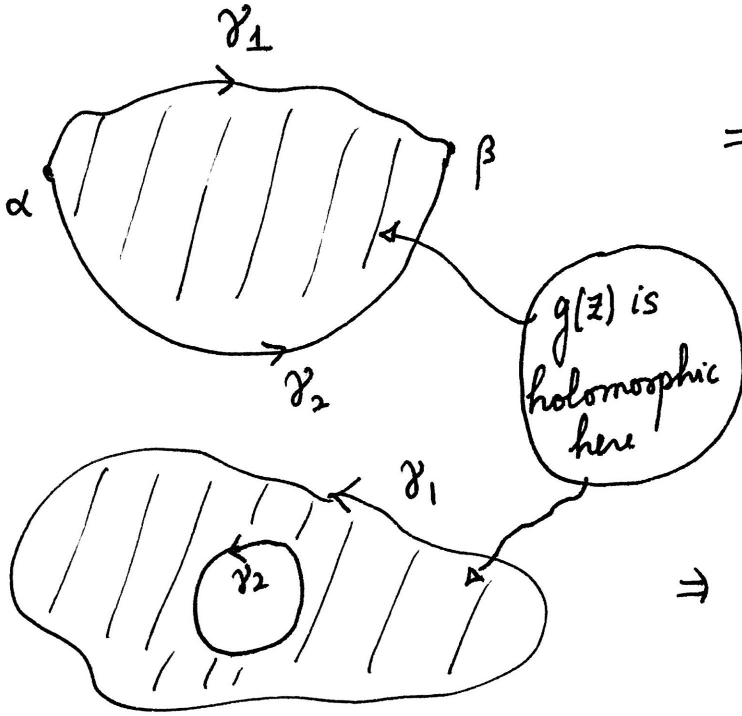
- <u>Antiderivative Thm.</u> $\boxed{G'(z) = g(z)} \Rightarrow \boxed{\int_\gamma g(z)\,dz = G\left(\begin{smallmatrix}\text{end point}\\ \text{of } \gamma\end{smallmatrix}\right) - G\left(\begin{smallmatrix}\text{starting}\\ \text{point of } \gamma\end{smallmatrix}\right)}$.

- <u>Cauchy's Theorem</u> [γ is a contour (= simple and closed), g is holomorphic on Interior (γ).]

$$\boxed{\int_\gamma g(z)\,dz = 0}$$



- <u>Principle of contour deformation</u>

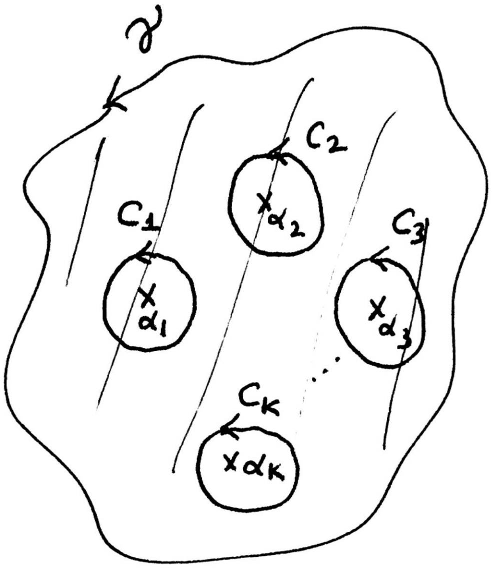


$$\Rightarrow \int_{\gamma_1} g(z)\,dz = \int_{\gamma_2} g(z)\,dz$$

$$\Rightarrow \int_{\gamma_1} g(z)\,dz = \int_{\gamma_2} g(z)\,dz$$

- If $g(z)$ is NOT holomorphic on the entire Interior(γ), but, say only on Interior$(\gamma) - \{\alpha_1, \alpha_2, \dots, \alpha_K\}$:

$$\int_\gamma g(z)\,dz \overset{\text{by Contour deformation}}{=} \sum_{j=1}^{K} \int_{C_j} g(z)\,dz$$



[$C_1, C_2, \dots, C_K$ are small contours around $\alpha_1, \dots, \alpha_K$; such that $\alpha_j \in$ Interior (C_j)
$\alpha_\ell \in$ Exterior (C_j) for $\ell \neq j$.]

$g(z)$ is holomorphic on Interior$(\gamma) - \{\alpha_1, \alpha_2, \dots, \alpha_K\}$

- Cauchy's Integral Formula

$$\boxed{\int_C \frac{f(z)}{(z-\alpha)^{n+1}}\,dz = \frac{2\pi i}{n!} f^{(n)}(\alpha)}$$

- C is a counterclockwise contour. f is holomorphic on Interior(C)
- $\alpha \in$ Interior(C) • $n \in \mathbb{Z}_{\geq 0}$.

(R2.4) Applications of Cauchy's integral formula:

- Liouville's Theorem. – Every entire, bounded holomorphic function is constant.

(means: domain $= \mathbb{C}$) (means: there exists $M \in \mathbb{R}_{\geq 0}$ such that $|f(z)| \leq M$; $\forall z \in \mathbb{C}$.)

- Fundamental Theorem of Algebra. – Every polynomial over $\mathbb{C}$ is a product of linear ones.

Remark. – The idea behind the proof of Liouville's theorem is important. Namely, we can use the principle of contour deformation, combined with the important inequality to conclude (in some cases) that $\int_C g(z)\,dz = 0$. (see Problem 2 of Practice Exam 2).

- Partial Fractions (Let $\alpha_1, \alpha_2, \ldots, \alpha_\ell \in \mathbb{C}$, $m_1, m_2, \ldots, m_\ell \in \mathbb{Z}_{\geq 1}$, $P(z) = a_n z^n + a_{n-1} z^{n-1} + \ldots + a_0$ polynomial of degree $n < m = m_1 + \ldots + m_\ell$)

$$g(z) = \frac{P(z)}{(z-\alpha_1)^{m_1} \cdots (z-\alpha_\ell)^{m_\ell}} = \frac{A_{1,1}}{z-\alpha_1} + \frac{A_{1,2}}{(z-\alpha_1)^2} + \ldots + \frac{A_{1,m_1}}{(z-\alpha_1)^{m_1}}$$

$$+ \frac{A_{2,1}}{z-\alpha_2} + \frac{A_{2,2}}{(z-\alpha_2)^2} + \ldots + \frac{A_{2,m_2}}{(z-\alpha_2)^{m_2}}$$

$$+ \ldots + \frac{A_{\ell,1}}{z-\alpha_\ell} + \frac{A_{\ell,2}}{(z-\alpha_\ell)^2} + \ldots + \frac{A_{\ell,m_\ell}}{(z-\alpha_\ell)^{m_\ell}}$$

Using Cauchy's integral formula

$$A_{j,p} = \frac{1}{2\pi i} \int_{C_j} (z-\alpha_j)^{p-1} g(z)\, dz \qquad \begin{pmatrix} 1 \le j \le l \\ 1 \le p \le m_j \end{pmatrix}$$

C_j is a small counterclockwise contour : $\begin{cases} \alpha_j \in \text{Interior}(C_j) \\ \alpha_k \in \text{Exterior}(C_j) \text{ for every } k \neq j. \end{cases}$

- Using the idea mentioned in the remark on the last page, we proved

$$\frac{1}{2\pi i} \int_{C} g(z)\, dz = \begin{cases} 0 & \text{if } n < m-1, \\ a_n & \text{if } n = m-1. \end{cases}$$

$\begin{pmatrix} C: \text{counterclockwise countour} \\ \alpha_1, \alpha_2, \ldots, \alpha_l \in \text{Interior}(C) \end{pmatrix}$

e.g. $$\frac{1}{2\pi i} \int_{C} \frac{2z^2+1}{(z-i)(z-3)^2}\, dz = 2.$$

C : any contour (counterclockwise) such that $i, 3 \in \text{Interior}(C)$