STUDY GUIDE FOR MID TERM 3

(S.1) Topics and where to find them.– Mid term 3 will consist of problems related to the following 5 topics:

Topic 1. Power series and radius of convergence.

(1) Review - what is, and how to compute, radius of convergence of a power series from Lecture 22, §22.3, 22.4.
(2) Go over some examples from §22.5, Problem Set 6 - problem 2.

Topic 2. Taylor and Laurent series.

(1) Review - Lectures 23, 24: read carefully statements of Theorem 23.4 and Theorem 24.3.
(2) Go over examples from §23.5, Problem set 6: 4,5,9,10.

Topic 3. Three types of singularities

(1) Review - Lecture 24: read §24.5, §24.6 (also Lecture 25: §25.5).
(2) Go over some examples: §24.7. Problem 11 of Problem set 6.

Topic 4. Residue

(1) Review - read carefully Lecture 25: §25.6 and §25.7.
(2) Go over the examples from §25.8, Problem 1 of Problem Set 7.

Topic 5. Application to real integrals [1]

(1) Review - three types of real integrals: Class I and II (read Lecture 27: §27.1 carefully). Class III (read the paragraph from about Class III in Lecture 28: §28.0; then go over the statement of Jordan's Lemma §28.1).
(2) Go over the examples given in: Lecture 27, §27.2–§27.5 (for Class I and II). See example in Lecture 28, §28.2 (for Class III). Problems 7-10 of Set 7.

(S.2) Some helpful facts.– After you are done with (S.1), read the following list of important facts to keep in mind. If you have done (S.1) thoroughly, these results should not surprise you.

1. Some useful power series

(1) $e^z = \displaystyle\sum_{n=0}^{\infty} \frac{z^n}{n!}$, for every $z \in \mathbb{C}$.
(2) $\sin(z) = \displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!}$, for every $z \in \mathbb{C}$.
(3) $\cos(z) = \displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!}$, for every $z \in \mathbb{C}$.

[1] The problems from this topic tend to be lengthy and computational, which is a good news, because I cannot ask you too many/too complicated problems of this genre in the exam.

(4) $\ln(1+z) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{z^n}{n}$, for $|z| < 1$.

(5) $\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n$, for $|z| < 1$.

(6) More generally, $\frac{1}{(1-z)^{\ell+1}} = \sum_{n=0}^{\infty} \binom{n+\ell}{\ell} z^n$, for $|z| < 1$.

2. Facts about power series

(1) Power series can be differentiated and integrated term–wise (within their radius of convergence).

(2) Power series can be manipulated like polynomials (multiplication, addition, clearing denominator etc. are allowed operations).

3. Facts about singularities and residues. An isolated singularity of $f(z)$ at $\alpha \in \mathbb{C}$: means f may not be defined at α, and we can draw a tiny circle C around α such that α is the only singularity within (and including) C.

(1) It is often more convenient to deal with singularity at 0. Reason - the series expansions written above are all centered at 0. So, it will be helpful to keep in mind that "singularity of $f(z)$ at $z = \alpha$" is same as "singularity of $f(x+\alpha)$ at $x = 0$". This is nothing but a simple change of variables: $z = x + \alpha$.

$$\boxed{\operatorname*{Res}_{z=\alpha}(f(z)) = \operatorname*{Res}_{x=0}(f(x+\alpha))}$$

(2) To determine the order of pole at $z = 0$, simply replace each function involved with the first few terms of its Taylor series expansion. If we have a term z^N in the denominator, and everything else can be evaluated at $z = 0$ giving a non–zero answer, then N is the order of the pole. (If $N = 0$ - we are talking about a removable singularity).

(3) At this point, if N is the order of the pole of $f(z)$ at $z = 0$, then we just multiply and divide $f(z)$ by z^N:

$$f(z) = \frac{1}{z^N} \cdot \boxed{z^N f(z)} = \frac{1}{z^N} \cdot \varphi(z)$$

($\varphi(z)$ is the function in the box). There are only two ways to determine the residue (try in this order - if computing derivatives is turning into a tedious mess, use the series expansion method):

- $\operatorname*{Res}_{z=0}(f(z)) = \frac{\varphi^{(N-1)}(0)}{(N-1)!}$.
- $\operatorname*{Res}_{z=0}(f(z))$ = Coefficient of z^{N-1} in the Taylor series expansion of $\varphi(z)$ near $z = 0$.

(S.3) Real integrals.– Three types of real integrals are of interest here.

(1) $\int_0^{2\pi} R(\cos(\theta), \sin(\theta))\, d\theta$. This involves change of variables $z = e^{\mathbf{i}\theta}$.

(2) $\int_{-\infty}^{\infty} Q(x)\, dx$ (where $Q(x)$ is an even function). This involves (i) Integrating $Q(z)$ over C_R: consisting of line μ_R joining $-R$ to R; and semicircle in the upper half of the

complex plane γ_R joining R to $-R$. (ii) Making sure $\int_{\gamma_R} Q(z)\,dz$ goes to 0 as $R \to \infty$.

(3) $\int_{-\infty}^{\infty} (\cos(mx) \text{ or } \sin(mx))Q(x)\,dx$. This involves integrating $e^{\mathbf{i}mz}Q(z)$ over C_R as before. Jordan's lemma helps us to conclude $\int_{\gamma_R} e^{\mathbf{i}mz}Q(z)\,dz \to 0$ as $R \to \infty$. Real part of our answer corresponds to the problems with $\cos(mx)$ and imaginary part to $\sin(mx)$.

(S.4) Sample problems.– Now try to solve these problems. Their solutions are given in the next section, but give each a fair shot before looking at the solutions.

(1) Prove that $\sum_{n=0}^{\infty} n^2 z^n = \dfrac{z(1+z)}{(1-z)^3}$, for $|z| < 1$.

(2) Write the Taylor series expansion of $\dfrac{1}{z^2+z+1}$ near $z = 0$. What is its radius of convergence?

(3) Write the Laurent series expansion of $\dfrac{1}{z(z-3)^2}$ near $z = 3$.

(4) Let C be the counterclockwise circle of radius 3 centered at 2. Prove that $\int_C \dfrac{dz}{z^3(z-4)} = 0$.

(5) List the singularities of $\dfrac{1}{z^2(e^z-1)}$. Determine the order of the pole and the residue at $z = 0$.

(6) Compute $\int_0^{2\pi} \dfrac{d\theta}{2+\cos(\theta)}$.

(7) Compute $\int_{-\infty}^{\infty} \dfrac{\cos(2x)}{x^2+9}\,dx$.

(S.5) Solutions.–

(1) Let us start from the geometric series

$$\sum_{n=0}^{\infty} z^n = \frac{1}{1-z}, \text{ for } |z| < 1$$

Take its derivative to get $\sum_{n=0}^{\infty} nz^{n-1} = \frac{1}{(1-z)^2}$. Now multiply it by z so that $\sum_{n=0}^{\infty} nz^n = \frac{z}{(1-z)^2}$. Taking derivative again and multiplying by z gives us:

$$\sum_{n=0}^{\infty} n^2 z^n = z \cdot \frac{d}{dz}\left(\frac{z}{(1-z)^2}\right) = z \cdot \left(\frac{(1-z)^2 + 2z(1-z)}{(1-z)^4}\right) = z \cdot \frac{1+z}{(1-z)^3}.$$

Since taking derivative preserves the radius of convergence, the last equation is valid on $|z| < 1$.

(2) *Solution 1.* $z^2 + z + 1 = 0$ has two solutions: $\alpha = \frac{-1+\mathbf{i}\sqrt{3}}{2}$ and $\beta = \frac{-1-\mathbf{i}\sqrt{3}}{2}$. Writing the partial fractions gives us:

$$\frac{1}{z^2+z+1} = \frac{1}{(z-\alpha)(z-\beta)} = \frac{1}{\alpha-\beta} \cdot \left(\frac{1}{z-\alpha} - \frac{1}{z-\beta}\right)$$

Using geometric series again:

$$\frac{1}{z-t} = -\frac{1}{t} \cdot \frac{1}{1-\frac{z}{t}} = -\sum_{n=0}^{\infty} \frac{z^n}{t^{n+1}}, \text{ valid on } |z| < |t|.$$

$$\frac{1}{z^2+z+1} = \frac{1}{\alpha-\beta} \sum_{n=0}^{\infty} \left(\frac{1}{\beta^{n+1}} - \frac{1}{\alpha^{n+1}}\right) z^n.$$

This is the Taylor series expansion of $\frac{1}{(z-\alpha)(z-\beta)}$, centered at 0. Therefore, its radius of convergence is the distance between 0 and the nearest singularity: α or β. As both $|\alpha| = |\beta| = 1$, so the radius of convergence is 1.

Solution 2. Multiply and divide by $1-z$ to get:

$$\frac{1}{z^2+z+1} = \frac{1-z}{(1-z)(1+z+z^2)} = \frac{1-z}{1-z^3} = (1-z) \cdot \sum_{n=0}^{\infty} z^{3n}, \quad \text{for } |z|^3 < 1.$$

Hence the Taylor series is $\sum_{n=0}^{\infty}(z^{3n} - z^{3n+1}) = 1 - z + z^3 - z^4 + z^6 - z^7 + \cdots$ with radius of convergence $= 1$.

(3) Laurent series expansion of $\frac{1}{z(z-3)^2}$.

$$\frac{1}{z} = \frac{1}{(z-3)+3} = \frac{1}{3} \cdot \frac{1}{1 - \frac{-(z-3)}{3}} = \frac{1}{3}\sum_{n=0}^{\infty} (-1)^n \frac{(z-3)^n}{3^n}.$$

Hence, $\dfrac{1}{z(z-3)^2} = \dfrac{1}{(z-3)^2}\sum_{n=0}^{\infty}(-1)^n\dfrac{(z-3)^n}{3^{n+1}}$.

(4) *(Direct computation.)* C encloses both the singularities 0 and 4 of the integrand. Let C_1 be a small counterclockwise circle around 0 and C_2 around 4. Then,

$$\int_C \frac{dz}{z^3(z-4)} = \int_{C_1}\frac{dz}{z^3(z-4)} + \int_{C_2}\frac{dz}{z^3(z-4)}.$$

$$\int_{C_1}\frac{dz}{z^3(z-4)} = 2\pi\mathbf{i}\frac{1}{2}\left[\frac{d^2}{dz^2}\frac{1}{z-4}\right]_{z=0} = \pi\mathbf{i}[2(z-4)^{-3}]_{z=0} = -\frac{\pi\mathbf{i}}{32}.$$

$$\int_{C_2}\frac{dz}{z^3(z-4)} = 2\pi\mathbf{i}[1/z^3]_{z=4} = \frac{\pi\mathbf{i}}{32}.$$

Hence, their sum is 0.

(Indirectly) Since C encloses all the singularities, we have

$$\int_C \frac{dz}{z^3(z-4)} = -2\pi\mathbf{i}\operatorname*{Res}_{z=\infty}\left(\frac{1}{z^3(z-4)}\right) = 2\pi\mathbf{i}\operatorname*{Res}_{w=0}\left(\frac{w^{-2}}{w^{-3}(w^{-1}-4)}\right) = 2\pi\mathbf{i}\operatorname*{Res}_{w=0}\left(\frac{w^2}{1-4w}\right)$$

which is 0, since $\dfrac{w^2}{1-4w}$ is holomorphic at $w=0$ [2].

(5) The singularities of $\dfrac{1}{z^2(e^z-1)}$ are the points where the denominator becomes 0. Meaning, $z=0$ and $z=2\pi n\mathbf{i}$, where $n\in\mathbb{Z}_{\neq 0}$.

Near $z=0$, we have: $e^z-1 = z+\dfrac{z^2}{2!}+\cdots$ Therefore,

$$\frac{1}{z^2(e^z-1)} = \frac{1}{z^3\left(1+\frac{z}{2!}+\cdots\right)}$$

Hence the pole at $z=0$ is of order 3. So, let us write:

$$\frac{1}{z^2(e^z-1)} = \frac{1}{z^3}\cdot\boxed{\frac{z}{e^z-1}}$$

Then, $\operatorname*{Res}_{z=0}\left(\dfrac{1}{z^2(e^z-1)}\right)$ is the coefficient of z^2 in the Taylor series expansion of $\dfrac{z}{e^z-1}$.

$$\frac{z}{e^z-1} = b_0+b_1z+b_2z^2+\cdots \Rightarrow z = \left(z+\frac{z^2}{2}+\frac{z^3}{6}+\cdots\right)\left(b_0+b_1z+b_2z^2+\cdots\right)$$

Comparing coefficients of z gives us $1=b_0$; coefficients of z^2 gives us $0=b_1+\dfrac{b_0}{2} \Rightarrow b_1=-\dfrac{1}{2}$; and finally z^3 gives:

$$0 = b_2+\frac{b_1}{2}+\frac{b_0}{6} \Rightarrow b_2 = \frac{1}{12}.$$

[2] Holomorphic functions have zero residue, but zero residue does not imply holomorphic. For example, $\operatorname*{Res}_{z=0}\left(z^{-2}\right)=0$

Hence, $\operatorname*{Res}_{z=0}\left(\frac{1}{z^2(e^z-1)}\right) = \frac{1}{12}$.

(6) Change of variables: $z = e^{\mathbf{i}\theta}$ implies:

$$\cos(\theta) = \frac{z+z^{-1}}{2} \qquad \text{and} \qquad d\theta = \frac{dz}{\mathbf{i}z}.$$

Let C be the counterclockwise circle of radius 1, centered at 0. Then,

$$\int_0^{2\pi} \frac{d\theta}{2+\cos(\theta)} = \int_C \frac{1}{2+\frac{z+z^{-1}}{2}} \frac{dz}{\mathbf{i}z} = \frac{1}{\mathbf{i}} \int_C \frac{2}{z^2+4z+1}\, dz\ .$$

Now, $z^2+4z+1=0$ has two roots:

$$\alpha_1 = -2+\sqrt{3} \qquad \text{and} \qquad \alpha_2 = -2-\sqrt{3}$$

While $|\alpha_2| = 2+\sqrt{3} > 1$, the fact that $\alpha_1\alpha_2 = 1$ implies that $|\alpha_1| < 1$, hence α_1 is within C and α_2 is outside of C. Applying Cauchy's integral formula:

$$\frac{2}{\mathbf{i}} \int_C \frac{dz}{(z-\alpha_1)(z-\alpha_2)} = \frac{2}{\mathbf{i}} 2\pi\mathbf{i} \frac{1}{\alpha_1-\alpha_2} = 4\pi \frac{1}{2\sqrt{3}} = \frac{2\pi}{\sqrt{3}}.$$

(7) We are going to integrate $\frac{e^{2\mathbf{i}z}}{z^2+9}$ over the contour C_R (see figure below).

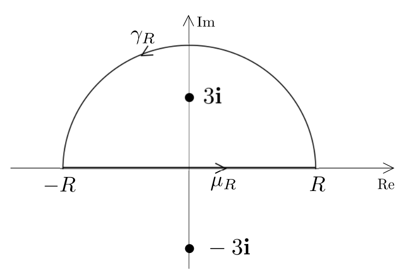

FIGURE 1. Contour C_R consisting of two smooth parts.

$$\int_{-\infty}^{\infty} \frac{\cos(2x)}{x^2+9}\, dx = \operatorname{Re}\left(\lim_{R\to\infty}\left(\int_{C_R} \frac{e^{2\mathbf{i}z}}{z^2+9}\, dz - \int_{\gamma_R} \frac{e^{2\mathbf{i}z}}{z^2+9}\, dz\right)\right)$$

Step 1. Integral over C_R, to be computed using Cauchy's integral formula:

$$\int_{C_R} \frac{e^{2\mathbf{i}z}}{(z+3\mathbf{i})(z-3\mathbf{i})}\, dz = 2\pi\mathbf{i}\left[\frac{e^{2\mathbf{i}z}}{z+3\mathbf{i}}\right]_{z=3\mathbf{i}} = 2\pi\mathbf{i}\frac{e^{-6}}{6\mathbf{i}} = \frac{\pi}{3}e^{-6}.$$

Step 2. Application of Jordan's lemma. Over γ_R, we have the following bound on $\frac{1}{z^2+9}$:

$$\left|\frac{1}{z^2+9}\right| \le \frac{1}{R^2-9} \to 0 \text{ as } R\to\infty.$$

Hence Jordan's lemma applies and we get $\lim_{R\to\infty} \int_{\gamma_R} \frac{e^{2\mathbf{i}z}}{z^2+9}\, dz = 0$.

Final step. Put everything together:

$$\int_{-\infty}^{\infty} \frac{\cos(2x)}{x^2+9}\,dx = \operatorname{Re}\left(\lim_{R\to\infty}\left(\int_{C_R} \frac{e^{2\mathbf{i}z}}{z^2+9}\,dz - \int_{\gamma_R} \frac{e^{2\mathbf{i}z}}{z^2+9}\,dz\right)\right) = \frac{\pi}{3}e^{-6}.$$