PROBLEMS ABOUT DOUBLY–PERIODIC FUNCTIONS.

Summary of results about doubly–periodic functions

Fix $\tau \in \mathbb{C}$ such that $\mathrm{Im}(\tau) > 0$.

We say $f(z)$ is a doubly–periodic function (with periods 1 and τ) if $f : \mathbb{C} \dashrightarrow \mathbb{C}$ is a meromorphic function and $f(z+1) = f(z)$ and $f(z+\tau) = f(z)$.

(1) Doubly–periodic + Holomorphic $\Rightarrow$ Constant.

(2) Sum of residues of $f(z)$ at poles lying within a fundamental parallelogram is zero.

(3) Number of zeroes of $f(z)$ = Number of poles of $f(z)$ within a fundamental parallelogram. The zeroes (and poles) are to be counted according to their order of vanishing (and the order of poles).

(4) Sum of zeroes - Sum of poles $= m + n\tau$, for some $m, n \in \mathbb{Z}$.

Problem 1. Let us take $\tau = \mathbf{i}$ and choose a fundamental parallelogram R to be the square with vertices $0, 1, \mathbf{i}, 1+\mathbf{i}$.

(Q1) Can there be a doubly–periodic function with only one zero within R, of order of vanishing 1, say at $\frac{1}{2} + \frac{1}{2}\mathbf{i}$?

(Q2) Can there be a doubly–periodic function with only one pole within R, of order 1?

(Q3) Can there be a doubly–periodic function $g(z)$ satisfying the following condition? Within R: (i) $g(z)$ has one pole of order 2 at $\beta = \frac{1}{2} + \frac{1}{2}\mathbf{i}$, (ii) $g(z)$ has two zeroes, each with order of vanishing 1, at $\alpha_1 = \frac{1}{8} + \frac{1}{4}\mathbf{i}$ and $\alpha_2 = \frac{7}{8} + \frac{3}{4}\mathbf{i}$.
If yes, prove that there is only one such function, upto multiplication by a constant. Use theta function (see next section) to write a formula for $g(z)$.

Problem 2. Why can't we have triply–periodic functions? Let $\tau \in \mathbb{C}$, with $\mathrm{Im}(\tau) > 0$ be fixed as before. Assume that $h \in \mathbb{C}$ is such that $\mathbb{Z}h \cap (\mathbb{Z} + \tau\mathbb{Z}) = \{0\}$ (that is, for any non–zero integer $p \in \mathbb{Z}_{\neq 0}$, ph cannot be written as $m + n\tau$, with $m, n \in \mathbb{Z}$). Assume that $f(z)$ is a meromorphic function, such that

$$f(z+1) = f(z), \qquad f(z+\tau) = f(z) \qquad \text{and} \qquad f(z+h) = f(z).$$

Prove that $f(z)$ cannot have any singularities. Deduce from it that $f(z)$ must be a constant.

Problem 3. Assume that $f(z)$ is a holomorphic function, satisfying $f(z+1) = f(z)$ and $f(z+\tau) = e^{-2\pi \mathbf{i} z} f(z)$. Let R be a fundamental parallelogram such that $f(z)$ does not have any zeroes on the boundary of R. Prove that $f(z)$ has exactly one zero within R, of order of vanishing 1.

Theta function

$\theta : \mathbb{C} \to \mathbb{C}$ is a holomorphic function defined on the entire complex plane, satisfying the following three properties:

(1) $\theta(z+1) = -\theta(z)$ and $\theta(z+\tau) = -e^{-\pi \mathbf{i}\tau} e^{-2\pi \mathbf{i} z}\theta(z)$.

(2) $\theta(z) = 0$ if, and only if $z = m + n\tau$ where $m, n \in \mathbb{Z}$. The order of vanishing of $\theta(z)$ at $z = m + n\tau$ is 1.

(3) $\theta'(0) = 1$.

In addition, $\theta(-z) = -\theta(z)$ (i.e, theta function is odd).

Explicitly, we have:

$$\theta(z) = \frac{\sin(\pi z)}{\pi} \cdot \prod_{n=1}^{\infty} \frac{\left(1 - e^{2\pi \mathbf{i} n\tau} e^{2\pi \mathbf{i} z}\right)\left(1 - e^{2\pi \mathbf{i} n\tau} e^{-2\pi \mathbf{i} z}\right)}{\left(1 - e^{2\pi \mathbf{i} n\tau}\right)^2}$$

$$\theta(z) = C \sum_{\ell \in \mathbb{Z}} (-1)^{\ell} e^{\pi \mathbf{i}\tau \ell(\ell+1)} e^{\pi \mathbf{i}(2\ell+1)z}$$

$$= C \sum_{k=0}^{\infty} (-1)^k e^{\pi \mathbf{i}\tau k(k+1)} \left(e^{(2k+1)\pi \mathbf{i} z} - e^{-(2k+1)\pi \mathbf{i} z}\right)$$

Here, C is a constant (depending on τ, but not on z). We didn't prove it, but its value is given by:

$$\frac{1}{C} = 2\pi \mathbf{i} \sum_{k=0}^{\infty} (-1)^k (2k+1) e^{\pi \mathbf{i}\tau k(k+1)} = 2\pi \mathbf{i} \prod_{n=1}^{\infty} \left(1 - e^{2\pi \mathbf{i} n\tau}\right)^3.$$

Problem 4. Prove that $\displaystyle\lim_{\mathrm{Im}(\tau)\to\infty} \theta(z) = \frac{e^{\pi \mathbf{i} z} - e^{-\pi \mathbf{i} z}}{2\pi \mathbf{i}}$.

Problem 5. Consider the function $\theta_1(z;\tau) = \displaystyle\sum_{\ell \in \mathbb{Z}} (-1)^{\ell} e^{\pi \mathbf{i}\ell(\ell+1)} e^{(2\ell+1)\pi \mathbf{i} z}$. Verify that $\theta_1(z;\tau)$ satisfies the following differential equation:

$$\frac{1}{(\pi \mathbf{i})^2} \frac{\partial^2 \theta_1}{\partial z^2} - \frac{4}{\pi \mathbf{i}} \frac{\partial \theta_1}{\partial \tau} = \theta_1(z;\tau)$$

Problem 6. Let $a \in \mathbb{C}$ be such that $a \notin \mathbb{Z} + \tau\mathbb{Z}$ (that is, a cannot be written as $m + n\tau$ with $m, n \in \mathbb{Z}$). Consider the following system of equations:

$$f(z+1) = f(z) \qquad \text{and} \qquad f(z+\tau) = e^{2\pi \mathbf{i} a} f(z).$$

(a) Use theta function to find a function $f(z)$ satisfying these equations.

(b) Verify that if f_1 and f_2 are two solutions of these equations, then their ratio must be doubly–periodic.

(c) Use (a) and (b) to prove that if these equations do not admit a holomorphic solution.

Problem 7. Let $x, y \in \mathbb{C}$ be fixed. Prove that $\dfrac{A\theta(z-x)\theta(z+x) + B\theta(z-y)\theta(z+y)}{\theta(z)^2}$ is doubly–periodic, for any constants $A, B \in \mathbb{C}$. Determine the values of A, B which will make this function holomorphic, and hence a constant C. Find the value of C.

Solutions

Problem 1. The answer to (Q2) is negative. Let $f(z)$ be a doubly–periodic function with only pole, say at $z = a$, of order 1 within a fundamental parallelogram R. Since the sum of its residues at poles within R has to be zero, $\operatorname{Res}_{z=a}(f(z)) = 0$. But that means a is not a pole of $f(z)$.

The answer to (Q1) is also negative. If $f(z)$ were to have only one zero, of order of vanishing 1 within R, then it will also have only one pole of order 1 within R. We already saw that it is impossible.

The answer to (Q3) is positive, since all the conditions check out: number of poles = number of zeroes = 2. Sum of poles - Sum of zeroes = 0 which is of the form $m + n\tau$ (with $m = n = 0$.) One such function can be written as: $g(z) = \dfrac{\theta(z-\alpha_1)\theta(z-\alpha_2)}{\theta(z-\beta)^2}$. Uniqueness, up to a constant, follows from Theorem 34.5.

Problem 2. Let $f(z)$ be a "triply–periodic" function, as in the problem. Let R be the parallelogram with vertices $0, 1, \tau, 1+\tau$.

If $f(z)$ has a pole, say α, within R, then by the third periodicity, $\alpha + ph$ is also a pole of $f(z)$, for every $p \in \mathbb{Z}$. By our assumption that ph cannot be written as $m + n\tau$, we get infinitely many poles of $f(z)$ which lie within R (for each $p \in \mathbb{Z}_{\neq 0}$, shift $\alpha + ph$ by 1 and τ to find α_p within R such that $\alpha + ph \equiv \alpha_p$ modulo $\Lambda_\tau = \mathbb{Z} + \tau\mathbb{Z}$.)

Since R is closed and bounded, these infinitely many poles will cluster around a point of R, creating an essential singularity, which contradicts the hypothesis that $f(z)$ is meromorphic.

Hence $f(z)$ cannot have any poles, so it is holomorphic and doubly–periodic, therefore a constant.

Problem 3. You should read the argument on page 6 of Lecture 35 for this problem.

Problem 4. Using the formula of $\theta(z)$ given above, and the fact that $\lim_{\mathrm{Im}(\tau)\to\infty} e^{2\pi\mathbf{i}n\tau} = 0$ for every $n \in \mathbb{Z}_{\geq 1}$, we get:

$$\lim_{\mathrm{Im}(\tau)\to\infty} \theta(z) = \frac{\sin(\pi z)}{\pi} = \frac{e^{\pi\mathbf{i}z} - e^{-\pi\mathbf{i}z}}{2\pi\mathbf{i}}.$$

Problem 5. The following equations are easy to verify:

$$\frac{1}{(\pi\mathbf{i})^2}\partial_z^2\theta_1 = \sum_{\ell\in\mathbb{Z}}(-1)^\ell e^{\pi\mathbf{i}\ell(\ell+1)}(2\ell+1)^2 e^{\pi\mathbf{i}(2\ell+1)z}$$

$$\frac{1}{\pi\mathbf{i}}\partial_\tau\theta_1 = \sum_{\ell\in\mathbb{Z}}(-1)^\ell e^{\pi\mathbf{i}\ell(\ell+1)}\ell(\ell+1) e^{\pi\mathbf{i}(2\ell+1)z}$$

The claimed differential equation now follows from: $(2\ell+1)^2 - 4\ell(\ell+1) = 1$.

Problem 6. (a) A solution is given by $f(z) = \dfrac{\theta(z-a)}{\theta(z)}$:

$$f(z+1) = \frac{\theta(z+1-a)}{\theta(z+1)} = \frac{-\theta(z-a)}{-\theta(z)} = f(z)$$

$$f(z+\tau) = \frac{\theta(z+\tau-a)}{\theta(z+\tau)} = \frac{-e^{-\pi\mathbf{i}\tau}e^{-2\pi\mathbf{i}(z-a)}\theta(z-a)}{-e^{-\pi\mathbf{i}\tau}e^{-2\pi\mathbf{i}z}\theta(z)} = e^{2\pi\mathbf{i}a}f(z)$$

(b) is trivial to check.

(c) If $g(z)$ is a holomorphic solution to these equations, and $f(z)$ is the solution given above, then $\dfrac{g(z)}{f(z)}$ is doubly–periodic, with only one pole within a fundamental parallelogram, at $z = a$, of order 1. In Problem 1 (Q2) we saw that there is no such function. Hence we cannot have a holomorphic solution.

Problem 7. Let $F(z) = \dfrac{A\theta(z-x)\theta(z+x) + B\theta(z-y)\theta(z+y)}{\theta(z)^2}$.

For any $c \in \mathbb{C}$, let $f(z) = \theta(z-c)\theta(z+c)$. Then, by the periodicity properties of $\theta(z)$, we have:

$$f(z+1) = f(z) \qquad \text{and} \qquad f(z+\tau) = e^{-2\pi\mathbf{i}\tau}e^{-4\pi\mathbf{i}z}f(z).$$

This implies that $F(z)$ is doubly–periodic.

Let us determine a choice of A, B which makes the numerator vanish at $z = 0$.

$$0 = A\theta(-x)\theta(x) + B\theta(-y)\theta(y) = -A\theta(x)^2 - B\theta(y)^2.$$

For instance, we can take $A = \theta(y)^2$ and $B = -\theta(x)^2$.

Now we know that $F(z) = \dfrac{\theta(y)^2\theta(z-x)\theta(z+x) - \theta(x)^2\theta(z-y)\theta(z+y)}{\theta(z)^2}$ is (i) doubly–periodic (ii) has no poles: if $F(z)$ were to have a pole, it must be at $z = 0$ (within a fundamental parallelogram containing 0), where the order is either 0 or 1 ($\theta(z)^2$ vanishes to order 2 at $z = 0$, and the numerator vanishes to order at least 1 at $z = 0$). The order of the pole at $z = 0$ cannot possibly be 1 (see Problem 1 (Q2)), so it must be 0 - i.e, $F(z)$ has no poles.

Being holomorphic and doubly–periodic $F(z) = C$ is a constant.

$$C = F(x) = \frac{-\theta(x)^2\theta(x-y)\theta(x+y)}{\theta(x)^2} = -\theta(x-y)\theta(x+y).$$

Thus, Problem 7 gives us a proof of the following identity:

$$\theta(y)^2\theta(z-x)\theta(z+x) - \theta(x)^2\theta(z-y)\theta(z+y) = -\theta(z)^2\theta(x-y)\theta(x+y)$$

Note that this identity can be obtained from Fay's trisecant identity (Lecture 35, §35.2) by setting $\alpha = z$, $\gamma = 0, \beta = x$ and $\delta = y$.