COMPLEX ANALYSIS:
PROBLEMS FOR $e^z, \sin(z), \cos(z), \ln(z), z^\alpha$.

Problem 1. Find all values of $z \in \mathbb{C}$ such that:
(a) $e^z = 1 + \sqrt{3}\,\mathbf{i}$. (b) $e^z = -2$.
(c) $e^z = 1 + \mathbf{i}$. (d) $e^{2z+1} = 1$.

Problem 2. Prove the following:

(1) $\left(\frac{e}{2}\left(-1-\sqrt{3}\,\mathbf{i}\right)\right)^{3\pi\mathbf{i}} = -e^{2\pi^2}$.

(2) $(1-\mathbf{i})^{4\mathbf{i}} = e^{\pi}\left(\cos(\ln(4)) + \mathbf{i}\sin(\ln(4))\right)$.

Problem 3. Prove that $\left|e^{-z}\right| < 1$ if, and only if $\mathrm{Re}(z) > 0$.

Problem 4. Prove that $\left|e^{z^2}\right| \le e^{|z|^2}$ for every $z \in \mathbb{C}$.

Problem 5. Compute the limit, as $z \to 0$, of the following functions.

$$\frac{\sin(z)}{z}, \qquad \frac{e^z - e^{-z}}{z}.$$

Problem 6. Identify the domains of the following functions. Then, compute their derivative with respect to z.
(a) e^{z^2+2z}. (b) $\ln(3z+1)$.
(c) $z^{1-\mathbf{i}}$. (d) $z\ln(z)$.
(e) $\left(z^2+1\right)^{\frac{1}{2}}$. (f) $\ln\left(z^2-1\right)$.

Problem 7. Use the definition of $\sin(z), \cos(z)$, for $z \in \mathbb{C}$, to prove the following trigonometric identities.

(1) $\cos(2z) = \cos^2(z) - \sin^2(z)$.

(2) $2\cos(z)\cos(w) = \cos(z+w) + \cos(z-w)$.

(3) $(1-\cos(z-w))(1-\cos(z+w)) = (\cos(z)-\cos(w))^2$.

Problem 8. (Optional, but fun!) Let $\mathcal{C}$ be a circle with center O. Let A and B two points on $\mathcal{C}$. Prove that, for any point C on $\mathcal{C}$ different from A and B, one of the following must hold (depending on whether ACB is the long, or the short arc):

$$\text{Either } \angle AOB = 2\angle ACB, \text{ or } \angle AOB = 2\left(\pi - \angle ACB\right).$$