COMPLEX ANALYSIS: PROBLEM SHEET 1

In the problems below, $\mathbb{C}$ is the set of complex numbers, $\mathbf{i}^2 = -1$, and $\overline{z}$ is the conjugate of $z \in \mathbb{C}$.

Problem 1. Compute the real and imaginary parts of the following complex numbers:
(a) $\dfrac{1}{\mathbf{i}}$, (b) $\dfrac{1-\mathbf{i}}{1+\mathbf{i}}$, (c) $\dfrac{2}{1-3\mathbf{i}}$, (d) $(1+\sqrt{3}\ \mathbf{i})^2$, (e) $(2-\mathbf{i})(3+\mathbf{i})$.

Problem 2. Compute the modulus and argument of the following complex numbers (make sure the argument you compute lies in the interval $(-\pi, \pi]$):
(a) $3\mathbf{i}$, (b) $2-5\mathbf{i}$, (c) $-1-\mathbf{i}$, (d) -2.

Problem 3. What is $\mathrm{Re}\left((1+\sqrt{3}\ \mathbf{i})^4\right)$?

Problem 4. Express $\cos(4\theta)$ in terms of $\cos(\theta)$ and $\sin(\theta)$. (*Hint: use de Moivre's formula*)

Problem 5. Solve the following quadratic equations.

(1) $z^2 - (1+\mathbf{i})z + \mathbf{i} = 0$.
(2) $z^2 + \mathbf{i}z + 1 = 0$.

Problem 6. Solve for all values of z in the following equations. Plot these roots in the complex plane.
(a) $z^3 = \mathbf{i}$, (b) $z^2 = 3+4\mathbf{i}$, (c) $z^8 = 1$, (d) $z^3 = -2+2\mathbf{i}$, (e) $z^4 = -4$.

Problem 7. Prove the following.

(1) For two complex numbers $z, w \in \mathbb{C}$: $\overline{zw} = \overline{z} \cdot \overline{w}$.
(2) For $z \in \mathbb{C}$: $\mathrm{Re}(z) = \dfrac{z+\overline{z}}{2}$, and $\mathrm{Im}(z) = \dfrac{z-\overline{z}}{2\mathbf{i}}$.
(3) For $z \neq 0$: $\left|\frac{1}{z}\right| = \frac{1}{|z|}$.

Problem 8. Prove the following identities:

(1) $|z+w|^2 = |z|^2 + |w|^2 + 2\,\mathrm{Re}(z\overline{w})$.
(2) $|z+w|^2 + |z-w|^2 = 2\left(|z|^2 + |w|^2\right)$.

Problem 9.

(1) For any complex number $\alpha \in \mathbb{C}$, prove that $-|\alpha| \leq \mathrm{Re}(\alpha) \leq |\alpha|$.
(2) Use the previous part, and problem 8 above, to prove the triangle inequality: $|z+w| \leq |z| + |w|$, for every $z, w \in \mathbb{C}$.

(3) Determine the conditions on z and w so that the triangle inequality becomes an equality.

Problem 10. Let $z \in \mathbb{C}$, $z \neq 1$ be such that $z^5 = 1$. Prove that $1+z+z^2+z^3+z^4 = 0$.

Problem 11. Let n be a positive integer, $n \geq 2$, and let $\omega = \cos\left(\frac{2\pi}{n}\right) + \sin\left(\frac{2\pi}{n}\right)\mathbf{i}$. Prove that $1 + \omega + \omega^2 + \ldots + \omega^{n-1} = 0$.

Problem 12. Verify the following identity, for any two real numbers a and b:

$$(1 - \cos(a-b))(1 - \cos(a+b)) = (\cos(a) - \cos(b))^2.$$

Problem 13. Let z and w be two non–zero complex numbers. Let θ be the angle that the line segment joining 0 and z forms with the line segment joining 0 and w. Prove that $\cos(\theta) = \frac{\mathrm{Re}(z\overline{w})}{|z||w|}$. (*Hint: think of the dot product of two vectors.*)

Problem 14. Sketch the subsets of the complex plane, described by the following:

(1) $\{z \in \mathbb{C} \text{ such that } |z-3| < 4\}$.
(2) $\{z \in \mathbb{C} \text{ such that } |z-2| + |z+2| = 5\}$.
(3) $\{z \in \mathbb{C} \text{ such that } \mathrm{Re}(z^2) = 1\}$.
(4) $\{z \in \mathbb{C} \text{ such that } \mathrm{Im}(z^2) = 2\}$.
(5) $\{z \in \mathbb{C} \text{ such that } 0 < \mathrm{Re}(\mathbf{i}z) < 1\}$.