## COMPLEX ANALYSIS: PROBLEM SHEET 2

**Problem 1.** For each of the following subsets of $\mathbb{C}$, determine whether it is open, closed, or neither.

(a) $|z - 2 + \mathbf{i}| \leq 1$ (b) $|2z + 3| > 4$

(c) $\mathrm{Im}\left(\frac{1}{z}\right) > 1 \ (z \neq 0)$ (d) $\mathrm{Im}(z) \neq 0$.

(e) $0 \leq \arg(z) \leq \frac{\pi}{2} \ (z \neq 0)$ (f) $\mathbb{C} \setminus \mathbb{Z} \ \ ( = \{z \in \mathbb{C} : z \text{ is not an integer}\})$

**Problem 2.** For the subsets given in Problem 1 above, determine whether they are bounded or not.

**Problem 3.** Sketch the following subset of $\mathbb{C}$ and determine whether it is connected or not: $\mathrm{Re}(z^2) > 1$.

**Problem 4.** In the following problems, a subset $S \subset \mathbb{C}$ and a point $z_0 \in \mathbb{C}$ are given. Determine whether $z_0$ is an accumulation point of $S$ or not.

(a) $S = \left\{\frac{1}{n} : n = 1, 2, 3 \ldots\right\}$, $z_0 = 0$.

(b) $S = \{0 \leq \arg(z) \leq \frac{\pi}{2}, z \neq 0\}$, $z_0 = 0$.

(c) $S = \mathbb{Z}$, $z_0 = \mathbf{i}$.

(d) $S = \{\mathrm{Im}(z) \neq 0\}$, $z_0 = 1$.

**Problem 5.** Let $S \subset \mathbb{C}$ be such that every accumulation point of $S$ is in $S$. Prove that $S$ is closed.

**Problem 6.** Let $S \subset \mathbb{C}$ be a finite subset: $S = \{z_1, z_2, \ldots, z_n\}$. Prove that $S$ has no accumulation points.

**Problem 7.** Compute the following limits. (*Use the results of Section 5.3, Lecture 5*).

(a) $\lim\limits_{z \to 0} \frac{z^2 - 2}{z + \mathbf{i}}$ (b) $\lim\limits_{z \to 1+\mathbf{i}} \frac{z + \mathbf{i}}{z^2 + 2}$

(c) $\lim\limits_{z \to -2} \frac{1}{z^3}$ (d) $\lim\limits_{z \to \mathbf{i}} \frac{\mathbf{i}z^3 - 1}{z + \mathbf{i}}$

**Problem 8.** Verify that the following limits do not exist: (*See the argument in Section 5.2, Lecture 5*). (a) $\lim\limits_{z \to 0} \frac{\overline{z}}{|z|}$, (b) $\lim\limits_{z \to 0} \frac{\mathrm{Im}(z^2)}{|z|^2}$.

*In problems 9 and 10, use Definition (4.3) from Lecture 4.*

**Problem 9.** Let $f(z)$ and $g(z)$ be two functions of a complex variable. Assume that the following two statements are true:

(1) $\lim_{z \to z_0} f(z) = 0$.

(2) There exists $M > 0$ and $r > 0$ such that $|g(z)| < M$ for every $z \in D^{\times}(z_0; r)$. Recall: $D^{\times}(\alpha; r) = \{w \in \mathbb{C} : 0 < |w - \alpha| < r\}$ is the punctured disc.

Prove that $\lim_{z \to z_0} f(z)g(z) = 0$.

**Problem 10.** Let $f(z)$ and $g(z)$ be two functions of a complex variable. Assume that the following two limits exist:

$$\lim_{z \to z_0} f(z) = A, \qquad \lim_{z \to z_0} g(z) = B.$$

Prove the following:

(1) $\lim_{z \to z_0} (f(z) + g(z)) = A + B$.

(2) $\lim_{z \to z_0} f(z)g(z) = AB$.

**Problem 11.** Let $f(z)$ be a function of a complex variable. Assume that the following limit exists: $\lim_{z \to z_0} f(z) = A$. Prove the following:

(1) $\lim_{z \to z_0} \overline{f(z)} = \overline{A}$.

(2) $\lim_{z \to z_0} |f(z)|^2 = |A|^2$.

**Problem 12.** For each of the following functions, check if the Cauchy–Riemann equations hold. Here $x = \mathrm{Re}(z)$ and $y = \mathrm{Im}(z)$.

(1) $f(z) = x^2 + y^2 + 2xy\mathbf{i}$.

(2) $f(z) = e^x \cos(y) + e^x \sin(y)\mathbf{i}$.

**Problem 13.** For each function $f(z)$ below, write $f(z) = u(x, y) + v(x, y)\mathbf{i}$, where $x = \mathrm{Re}(z)$ and $y = \mathrm{Im}(z)$. Check the Cauchy–Riemann equations.

(1) $f(z) = z^2 + \overline{z}$.

(2) $f(z) = \dfrac{1}{z}$.