COMPLEX ANALYSIS: PROBLEM SHEET 3

Problem 1. For the following functions, compute the derivative $f'(z)$.

(a) $f(z) = \dfrac{z}{z-1}$. (b) $f(z) = z^2 + 1 + z^{-2}$.

(c) $f(z) = \left(\dfrac{z+1}{z-\mathbf{i}}\right)^6$. (d) $f(z) = \dfrac{z^2+2z}{z^3}$.

(e) $f(z) = 3z^2 - 2z + 4$. (f) $f(z) = \left(2z^2 + \mathbf{i}\right)^5$.

Problem 2. For each of the functions below, verify that the Cauchy–Riemann equations hold. Then, compute the derivative ($f'(z) = u_x + v_x\mathbf{i} = v_y - u_y\mathbf{i}$.)

(a) $(2-y) + x\mathbf{i}$.
(b) $x^3 - 3xy^2 + (3x^2y - y^3)\mathbf{i}$.
(c) $\dfrac{x^2-y^2}{(x^2+y^2)^2} - \dfrac{2xy}{(x^2+y^2)^2}\mathbf{i}$.
(d) $e^x\cos(y) + e^x\sin(y)\mathbf{i}$.

Problem 3. Let $u(x,y) = x^2 - y^2 + x$.

(a) Verify that the Laplace equation holds for $u(x,y)$. *Recall: The Laplace equation for $u(x,y)$ is:* $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$.

(b) Compute a function $v(x,y)$ so that $f(z) = u(x,y) + v(x,y)\mathbf{i}$ is $\mathbb{C}$–differentiable.

(c) Write the function from the previous part in terms of z (and $\overline{z}$, but since it is $\mathbb{C}$–differentiable, there should not be any dependence on $\overline{z}$).

Problem 4. Let $u(x,y) = \dfrac{x}{x^2+y^2}$. Redo (a)–(c) of Problem 3 with this function.

Problem 5. Let $u(x,y) + v(x,y)\mathbf{i}$ be a $\mathbb{C}$–differentiable function. Prove that the following two vectors are orthogonal:

$$\vec{\nabla} u = \left\langle \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y} \right\rangle; \qquad \vec{\nabla} v = \left\langle \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y} \right\rangle.$$

This problem proves that, for any two real constants a, b, the level curves $u(x,y) = a$ and $v(x,y) = b$, wherever they meet, they meet at right angles.

Problem 6. Consider the change of variables from Cartesian to polar coordinates (valid on the open set $\mathbb{C}^\times := \mathbb{C} \setminus \{0\}$).

$$x = r\cos(\theta) \qquad \text{and} \qquad y = r\sin(\theta).$$

Verify that the Cauchy–Riemann equations in (r, θ) variables take the following form:

$$r\frac{\partial u}{\partial r} = \frac{\partial v}{\partial \theta} \qquad \text{and} \qquad \frac{\partial u}{\partial \theta} = -r\frac{\partial v}{\partial r}.$$

Problem 7. Use Problem 6 to verify that the following function is $\mathbb{C}$–differentiable, on the open set $\Omega = \{z \in \mathbb{C} : z \neq 0, -\pi < \arg(z) < \arg(z)\}$ (this is just the complex plane, with the negative real axis removed):

$$f(z) = \ln(r) + \theta\mathbf{i}.$$

Prove that $f'(z) = \dfrac{1}{z}$.

Problem 8. Prove that the Laplace equation for a real–valued function $g(x, y)$ of two real variables, takes the following form when written in polar coordinates (r, θ).

$$\frac{\partial^2 g}{\partial r^2} + \frac{1}{r}\frac{\partial g}{\partial r} + \frac{1}{r^2}\frac{\partial^2 g}{\partial \theta^2} = 0.$$

Problem 9. Use Problem 8 to prove that any solution $g(r, \theta)$ of the Laplace equation, which is independent of θ, has to be of the following form: $g(r, \theta) = C_1 \ln(r) + C_2$, where $C_1, C_2 \in \mathbb{R}$ are arbitrary constants.

Problem 10. Let $f(z) = u(x, y) + v(x, y)\mathbf{i}$ be a $\mathbb{C}$–differentiable function. Assume that $u(x, y) = C \in \mathbb{R}$ is a constant function. Prove that $f(z)$ is also a constant function.

Problem 11. Let $P(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n$ be a polynomial of degree n. Here $n \in \mathbb{Z}_{\geq 0}$ and $a_0, a_1, \ldots, a_n \in \mathbb{C}$ are fixed complex numbers, with $a_n \neq 0$. Prove that, for each $k = 0, 1, \ldots, n$, we have:

$$a_k = \frac{P^{(k)}(0)}{k!}.$$

$P^{(k)}(z)$ is the k^{th} derivative of $P(z)$:

$$P^{(0)}(z) = P(z), P^{(1)}(z) = P'(z), P^{(2)}(z) = P''(z), \ldots \text{ and so on.}$$

$k! = 1 \cdot 2 \cdot \ldots \cdot k$ is the factorial.