# COMPLEX ANALYSIS: PROBLEM SHEET 4

**Problem 1.** Let $\phi \in (0, 2\pi]$. Compute the length of the arc of the circle of radius $R$, centered at 0, between angles 0 to $\phi$. (*Use the parametrization:* $\gamma(\theta) = Re^{\mathbf{i}\theta}$, $0 \le \theta \le \phi$.)

**Problem 2.** Let $\gamma : [0, 1] \to \mathbb{C}$ be any piecewise smooth path joining 0 to $1 + 3\mathbf{i}$. Compute the following:
(a) $\displaystyle\int_\gamma \cos(z)\, dz$, (b) $\displaystyle\int_\gamma \sin(z^2) z\, dz$, (c) $\displaystyle\int_\gamma e^{2z}\, dz$,
(d) $\displaystyle\int_\gamma z^3 + 2\, dz$, (e) $\displaystyle\int_\gamma (z^2 + 1)^2\, dz$.

**Problem 3.** Let $k \in \mathbb{Z}$. Prove that:

$$\int_0^{2\pi} e^{\mathbf{i}k\theta}\, d\theta = \begin{cases} 0 & \text{if } k \neq 0, \\ 2\pi & \text{if } k = 0. \end{cases}$$

**Problem 4.** Let $f(z)$ be a continuous function defined on an open, connected subset $\Omega \subseteq \mathbb{C}$. Let $\gamma : [a, b] \to \Omega$ be a smooth path, and let $\tau : [0, 1] \to [a, b]$ be a smooth function. Define $\widetilde{\gamma} : [0, 1] \to \Omega$ as $\widetilde{\gamma}(t) = \gamma(\tau(t))$. Prove that

$$\int_{\widetilde{\gamma}} f(z)\, dz = \int_\gamma f(z)\, dz.$$

**In Problems 5-12 below: (a) sketch $\gamma$, (b) compute $\int_\gamma f(z)\, dz$.**

**Problem 5.** $f(z) = \mathrm{Re}(z)$, and $\gamma(t) = Re^{\mathbf{i}t}$, $0 \le t \le 2\pi$.

**Problem 6.** $f(z) = \mathrm{Im}(z)$, and $\gamma(t) = Re^{\mathbf{i}t}$, $0 \le t \le 2\pi$.

**Problem 7.** $f(z) = y - x - (3x^2)\mathbf{i}$, where $z = x + y\mathbf{i}$, and

$$\gamma(t) = \begin{cases} t\mathbf{i}, & \text{if } 0 \le t \le 1, \\ (t-1) + \mathbf{i}, & \text{if } 1 \le t \le 2, \\ (3-t)(1+\mathbf{i}), & \text{if } 2 \le t \le 3. \end{cases}$$

**Problem 8.** $f(z) = \overline{z}$ and $\gamma(\theta) = 2e^{\mathbf{i}\theta}$, $-\dfrac{\pi}{2} \le \theta \le \dfrac{\pi}{2}$.

**Problem 9.** $f(z) = 2\,\mathrm{Re}(z) + \mathrm{Im}(z)$, and $\gamma(t) = t(1 + \mathbf{i})$, $0 \le t \le 1$.

**Problem 10.** Let $n \in \mathbb{Z}$, $z_0 \in \mathbb{C}$, $R \in \mathbb{R}_{>0}$.

$$f(z) = (z - z_0)^{n-1}, \text{ and } \gamma(\theta) = z_0 + Re^{\mathbf{i}\theta}, 0 \le \theta \le 2\pi.$$

**Problem 11.** $f(z) = \frac{1}{z}$, and $\gamma(t) = 1 + t\mathbf{i}$, $-1 \leq t \leq 1$.

**Problem 12.** Let $r \in \mathbb{R}_{>0}$ and $\phi \in (-\pi, \pi)$.

$$f(z) = \frac{1}{z}, \text{ and } \gamma(t) = (1 + t(r-1))e^{\mathbf{i}t\phi}, \ 0 \leq t \leq 1.$$

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**Bonus.** Let $\gamma$ be a positively oriented contour. Let $S$ be the area of the interior of $\gamma$. Prove that

$$\int_\gamma \mathrm{Re}(z)\, dz = \mathbf{i}S, \text{ and } \int_\gamma \mathrm{Im}(z)\, dz = -S.$$

*This is a generalization of Problems 5,6 above. Hint: Green's theorem, or recall how area enclosed by a curve was defind in Calculus I/II.*