

## COMPLEX ANALYSIS: PROBLEM SHEET 5

**Problem 1.** Let  $C$  be the positively oriented contour, which is the boundary of the square with corners  $2 + 2\mathbf{i}$ ,  $2 - 2\mathbf{i}$ ,  $-2 - 2\mathbf{i}$ ,  $-2 + 2\mathbf{i}$ . Evaluate each of the following integrals.

$$\begin{aligned} \text{(a)} \quad & \int_C \frac{e^{-z}}{z - \frac{\pi\mathbf{i}}{2}} dz & \text{(b)} \quad & \int_C \frac{\cos(z)}{z(z^2 + 8)} dz & \text{(c)} \quad & \int_C \frac{z}{2z + 1} dz \\ \text{(d)} \quad & \int_C \frac{e^z + e^{-z}}{z^4} dz & \text{(e)} \quad & \int_C \frac{\sin(z)}{(z - 2)^2} dz. \end{aligned}$$

**Problem 2.** Let  $C$  be a counterclockwise oriented contour. Let us define:

$$f(w) = \int_C \frac{z^3 + 2z}{(z - w)^3} dz.$$

Prove that  $f(w) = 6\pi\mathbf{i}w$ , when  $w \in \text{Interior}(C)$ , and is 0, when  $w \in \text{Exterior}(C)$ .

**Problem 3.** Let  $C$  be the counterclockwise oriented circle of radius 1, centered at 0. Prove that, for any  $a \in \mathbb{R}$ ,

$$\int_C \frac{e^{az}}{z} dz = 2\pi\mathbf{i}.$$

Use the equation above to show that:

$$\int_0^\pi e^{a \cos(\theta)} \cos(a \sin(\theta)) d\theta = \pi.$$

**Problem 4.** Let  $\gamma(\theta) = e^{\mathbf{i}\theta}$ ,  $0 \leq \theta \leq 2\pi$ . Compute  $\int_\gamma \frac{e^z}{z^2(z^2 - 9)} dz$ .

**Problem 5.** Let  $f(z)$  be a holomorphic function defined on an open set  $\Omega \in \mathbb{C}$ . Let  $\alpha \in \Omega$ , and  $\rho \in \mathbb{R}_{\geq 0}$  be such that  $D(\alpha; \rho) \subset \Omega$ . (Recall:  $D(\alpha; \rho) = \{z \in \mathbb{C} : |z - \alpha| < \rho\}$ .) Prove that:

$$f(\alpha) = \frac{1}{2\pi} \int_0^{2\pi} f(\alpha + \rho e^{\mathbf{i}\theta}) d\theta.$$

**Problem 6.** Let  $C$  be the counterclockwise oriented circle of radius 4, centered at 0. Compute:  $\int_C \frac{1}{z^2 + 9} dz$ .

**Problem 7.** Let  $\gamma(t) = \frac{1}{2} - \frac{1}{2}e^{-it}$ ,  $0 \leq t \leq \pi$ . Compute  $\int_\gamma \frac{1}{1 + z^2} dz$ .

**Problem 8.** Let  $C$  be the counterclockwise oriented circle of radius  $\frac{1}{2}$ , centered at 1. Compute  $\int_C \frac{1}{z(z^2 - 1)} dz$ .

**Problem 9.** Let  $C$  be a counterclockwise oriented circle of radius  $> 1$ , centered at 0.

Compute  $\int_C \frac{z^2 + 2}{(z - 1)(z - \mathbf{i})^2} dz$ .

**Problem 10.** Let  $n \in \mathbb{Z}_{\geq 1}$  and consider the closed path  $\gamma_n : [0, 2\pi] \rightarrow \mathbb{C}$  given by:

$$\gamma_n(\theta) = e^{\mathbf{i}n\theta}.$$

Prove that  $\frac{1}{2\pi\mathbf{i}} \int_{\gamma_n} \frac{1}{z} dz = n$ .

**Problem 11.** Let  $C$  be the counterclockwise circle of radius 2, centered at 0. Compute

$$\int_C \frac{2z + 1}{z^2 + z + 1} dz.$$

**Problem 12.** Let  $R \in \mathbb{R}_{>0}$ , and let  $\gamma : [0, \pi] \rightarrow \mathbb{C}$ , given by  $\gamma(t) = Re^{\mathbf{i}t}$ , be the semicircle in the upper half plane, of radius  $R$ , centered at 0. Prove that:

$$\lim_{R \rightarrow \infty} \int_{\gamma} \frac{e^{\mathbf{i}z}}{z^2} dz = 0.$$

**Problem 13.** Compute the partial fraction decomposition of  $\frac{z^2 + 2}{(z - 1)(z - 3)^2}$ .

**Problem 14.** Consider the partial fraction decomposition:

$$\frac{3z^3 + 2z + \mathbf{i}}{z(z - 1)(z - 2)^2} = \frac{A}{z} + \frac{B}{z - 1} + \frac{C}{z - 2} + \frac{D}{(z - 2)^2}.$$

What is the value of  $C$  in the expression written above?