PROBLEM SHEET 6 - SOLUTIONS

Problem 1. Prove that $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges.

Solution 1. Use the integral test: $\sum_{n=2}^{\infty} \frac{1}{n^2} < \int_1^{\infty} \frac{1}{x^2}\, dx = 1$. Hence $\sum_{n=1}^{\infty} \frac{1}{n^2} < 1 + 1 = 2$ converges.

Solution 2. (Cauchy's criterion). Note the following inequalities:

$$\frac{1}{2^2} + \frac{1}{3^2} < \frac{2}{2^2} = \frac{1}{2}, \quad \frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{6^2} + \frac{1}{7^2} < \frac{4}{4^2} = \frac{1}{4}$$

Continuing this way, we have, for every $k \geq 1$:

$$\frac{1}{(2^k)^2} + \frac{1}{(2^k+1)^2} + \cdots + \frac{1}{(2^{k+1}-1)^2} < \frac{1}{2^k}.$$

So, for the infinite series:

$$\frac{1}{(2^k)^2} + \frac{1}{(2^k+1)^2} + \cdots < \frac{1}{2^k} + \frac{1}{2^{k+1}} + \cdots = \frac{1}{2^{k-1}}.$$

Now, given any $\varepsilon > 0$, take $\ell > 1$ large enough so that $\frac{1}{2^{\ell-1}} < \varepsilon$. Let $N = 2^\ell$. Then, for every $n \geq N$ and $p \geq 0$, we have:

$$\frac{1}{n^2} + \frac{1}{(n+1)^2} + \cdots + \frac{1}{(n+p)^2} < \frac{1}{N^2} + \frac{1}{(N+1)^2} + \cdots < \frac{1}{2^{\ell-1}} < \varepsilon.$$

So, Cauchy's criterion is met, and hence the series is convergent.

Remark. The exact value of this series is $\frac{\pi^2}{6}$. It was computed by Euler in 1734, using the infinite product expression of $\sin(z)$, which we will see in Lecture 29.

Problem 2. Find the radius of convergence of the following series.

(a) $\sum_{n=0}^{\infty} 7^n z^n$. Ratio of successive coefficients is 7. So radius of convergence is $\frac{1}{7}$.

(b) $\sum_{n=1}^{\infty} n^n z^n$. Limit of the ratio of successive coefficients:

$$\lim_{n\to\infty} \frac{(n+1)^{n+1}}{n^n} = \lim_{n\to\infty} (n+1) \cdot \left(1 + \frac{1}{n}\right)^n = \infty \cdot e = \infty$$

So, radius of convergence is 0.

(c) $\sum_{n=1}^{\infty} \frac{z^n}{n^2}$. Again, take the limit: $\lim_{n\to\infty} \frac{n^2}{(n+1)^2} = 1$. So, radius of convergence is 1.

Problem 3. Let $a, b, c \in \mathbb{C}$ be complex numbers such that $c \notin \mathbb{Z}_{\geq 0}$. The following power series is called *hypergeometric series*.

$$F(a,b;c;z) = 1 + \sum_{n=1}^{\infty} \frac{a(a+1)\cdots(a+n-1)b(b+1)\cdots(b+n-1)}{c(c+1)\cdots(c+n-1)} \frac{z^n}{n!}.$$

Prove that: (i) If $a, b \in \mathbb{Z}_{\leq 0}$, then the radius of convergence of $F(a,b;c;z)$ is ∞. (ii) If $a, b \notin \mathbb{Z}_{\leq 0}$, then its radius of convergence is 1.

Solution. (i) If either a or b is in $\mathbb{Z}_{\leq 0}$, the series is just a polynomial, hence has ∞ radius of convergence.

(ii) Assuming none of a, b, c are non–positive integers. We apply ratio test again: $\lim_{n\to\infty} \frac{(a+n)(b+n)}{(c+n)(n+1)} = 1$. Hence, radius of convergence is 1.

Problem 4. Prove that $\ln(1-z) = -\sum_{n=1}^{\infty} \frac{z^n}{n}$, for every $z \in D(0;1)$.

Solution. Since $\ln(1-z)$ is an antiderivative of $\frac{-1}{1-z} = -\sum_{k=0}^{\infty} z^k$, for $|z| < 1$, we get (by taking termwise antiderivative, which does not change the radius of convergence):

$$\ln(1-z) = C - \sum_{k=0}^{\infty} \frac{z^{k+1}}{k+1}, \text{ for } |z| < 1.$$

To fix the constant, set $z = 0$, to get $C = 0$.

Problem 5. For any $\ell \in \mathbb{Z}_{\geq 0}$, prove that:

$$\frac{1}{(1-z)^{\ell+1}} = \sum_{k=0}^{\infty} \binom{k+\ell}{\ell} z^k \text{ for every } z \in D(0;1).$$

Solution. We will prove it by induction on ℓ. The base case is $\ell = 0$: $\frac{1}{1-z} = \sum_{k=0}^{\infty} z^k$ for every $z \in D(0;1)$. This was already proved in Lecture 23, (23.2) Example.

Now assume that the statement is known to hold for $\ell = n \in \mathbb{Z}_{\geq 0}$:

$$\frac{1}{(1-z)^{n+1}} = \sum_{k=0}^{\infty} \binom{k+n}{n} z^k \text{ for every } z \in D(0;1).$$

Let us try to prove it for $n+1$. Take derivative of this equation to get (for every $z \in \mathbb{C}$ such that $|z| < 1$):

$$\frac{(n+1)}{(1-z)^{n+2}} = \sum_{k=1}^{\infty} k \frac{(n+k)!}{k!n!} z^{k-1} \;\Rightarrow\; \frac{1}{(1-z)^{n+2}} = \sum_{k=1}^{\infty} \frac{(n+k)!}{(k-1)!(n+1)!} z^{k-1}$$

Set $j = k - 1$ to get: $\dfrac{1}{(1-z)^{n+2}} = \sum_{j=0}^{\infty} \binom{n+j+1}{n+1} z^j$. The induction step holds, and finishes the proof.

Problem 6. Let $\sum_{k=0}^{\infty} a_k z^k$ be a power series with radius of convergence $R > 0$.

(1) Prove that the radius of convergence of $\sum_{k=0}^{\infty} a_k \dfrac{z^{k+1}}{k+1}$ is $\geq R$.

(2) Prove that the radius of convergence of $\sum_{k=0}^{\infty} a_k \dfrac{z^k}{k!}$ is ∞.

Solution 1 (ratio test). If $\dfrac{1}{R} = \lim_{n\to\infty} \dfrac{|a_{n+1}|}{|a_n|}$, then: $\lim_{n\to\infty} \dfrac{|a_{n+1}|}{n+1} \dfrac{n}{|a_n|} = \dfrac{1}{R}$, so, the series $\sum_{k=0}^{\infty} \dfrac{a_k}{k+1} z^k$ has radius of convergence R as well.

Similarly, $\lim_{n\to\infty} \dfrac{|a_{n+1}|}{(n+1)!} \dfrac{n!}{|a_n|} = \lim_{n\to\infty} \dfrac{|a_{n+1}|}{|a_n|} \dfrac{1}{n+1} = 0$. So, the radius of convergence of $\sum_{k=0}^{\infty} \dfrac{a_k}{k!} z^k$ is ∞.

Solution 2. $R > 0$ is the radius of convergence of $\sum_{k=0}^{\infty} a_k z^k$. This means:

- For any $0 \leq r < R$, we can find a constant M such that $|a_n| r^n < M$, for every $n \geq 0$.
- If $r > R$, then the sequence of numbers $\{|a_n| r^n\}_{n=0}^{\infty}$ is unbounded.

(1) Consider the series $\sum_{k=0}^{\infty} \dfrac{a_k}{k+1} z^k$. For any $0 \leq r < R$, let M be the constant so that $|a_n| r^n < M$ for every $n \geq 0$. Then:

$$\frac{|a_n|}{n+1} r^n < |a_n| r^n < M.$$

So the radius of convergence of the new series is at least R.

(2) Now consider the series $\sum_{k=0}^{\infty} \dfrac{a_k}{k!} z^k$. Fix a positive number $0 < \rho < R$, and take M_1 to be the constant as promised above, for ρ: $|a_n| \rho^n < M_1$ for every $n \geq 0$.

We want to show that the radius of convergence of the new series is ∞. Meaning, given any $r \in \mathbb{R}_{>0}$, we have to produce a constant M so that $\dfrac{|a_n|}{n!} r^n < M$, for every $n \geq 0$. We have:

$$\frac{|a_n|}{n!} r^n < \frac{M_1}{\rho^n} \frac{r^n}{n!} = M_1 \frac{t^n}{n!}, \text{ where } t = \frac{r}{\rho} \in \mathbb{R}_{>0}.$$

Since $\dfrac{t^n}{n!} \to 0$, as $n \to \infty$, we can find (using the definition of the limit, with $\varepsilon = 1$) $N > 0$ such that $\dfrac{t^n}{n!} < 1$ for every $n \geq N$. This gives us a bound on infinitely many terms: $\dfrac{|a_n|}{n!} r^n < M_1$, for every $n \geq N$. Now we just pick the largest number among the left–over finitely many terms:

$$M > \operatorname{Max}\left\{ |a_0|, \frac{|a_1|}{1!} r, \ldots, \frac{|a_{N-1}|}{(N-1)!} r^{N-1}, M_1 \right\} \ ,$$

so that $\dfrac{|a_n|}{n!} r^n < M$, for every $n \geq 0$.

Problem 7. Find the mistake in the following calculation.

$$\frac{1}{1-z} = \sum_{k=0}^{\infty} z^k \qquad \text{and} \qquad \frac{1}{1-z} = -\frac{1}{z} \cdot \frac{1}{1-z^{-1}} = -\sum_{\ell=0}^{\infty} z^{-\ell-1}.$$

Taking the difference we get: $0 = \displaystyle\sum_{k=0}^{\infty} z^k + \sum_{\ell=0}^{\infty} z^{-\ell-1}$.
Compare coefficient of (say) z to get $0 = 1$.

Solution. The first identity $\dfrac{1}{1-z} = \displaystyle\sum_{k=0}^{\infty} z^k$ holds for $|z| < 1$, while the second $\dfrac{1}{1-z} = -\displaystyle\sum_{\ell=0}^{\infty} z^{-\ell-1}$ for $|z| > 1$. Therefore, we cannot take the difference, since it will yield an identity which holds for *no complex number at all!*

Problem 8. Let $\displaystyle\sum_{k=0}^{\infty} \frac{2^k}{k} z^{-k}$ be a power series centered at ∞. What is its radius of convergence?

Solution. Again, by ratio test, the series converges for $z \in \mathbb{C}$ such that $|z^{-1}| < \frac{1}{2}$, that is, $|z| > 2$. According to the notation of Lecture 23, §23.6: $\{|z| > 2\} = D(\infty; 1/2)$, so the radius of convergence is $\frac{1}{2}$.

Problem 9. Compute the Taylor series expansion of $\cos(z)$, around 0, in the following two ways:

(1) By computing $\dfrac{d^n}{dz^n}(\cos(z))$ at $z = 0$.
Solution. $f(z) = \cos(z)$ gives $f'(z) = -\sin(z)$, $f''(z) = -\cos(z)$ and so on. Hence we get $f^{(2k+1)}(0) = 0$ and $f^{(2k)}(0) = (-1)^k$. So, the Taylor series expansion of $\cos(z)$ is: $\displaystyle\sum_{k=0}^{\infty} (-1)^k \frac{z^{2k}}{2k!}$.

(2) Using $\cos(z) = \dfrac{e^{\mathbf{i}z} + e^{-\mathbf{i}z}}{2}$ and the Taylor series of e^z.
Solution. Taylor series of the exponential function gives us:

$$e^{\mathbf{i}z} = 1 + \mathbf{i}z - \frac{z^2}{2!} - \mathbf{i}\frac{z^3}{3!} + \cdots \qquad \text{and} \qquad e^{-\mathbf{i}z} = 1 - \mathbf{i}z - \frac{z^2}{2!} + \mathbf{i}\frac{z^3}{3!} + \cdots$$

Hence, $\cos(z) = \dfrac{e^{\mathbf{i}z} + e^{-\mathbf{i}z}}{2} = 1 - \dfrac{z^2}{2!} + \dfrac{z^4}{4!} - \dfrac{z^6}{6!} + \cdots$

Problem 10. Compute the Taylor series expansion, and determine its radius of convergence:

(a) e^z centered at 1. $e^z = e^{(z-1)+1} = e \cdot \displaystyle\sum_{k=0}^{\infty} \frac{(z-1)^k}{k!}$. Radius of convergence $= \infty$, since e^z is defined on disc of any radius around 1.

(b) $\dfrac{z}{z^2+4}$ centered at 0. $\dfrac{z}{4}\dfrac{1}{1 - \frac{-z^2}{4}} = \displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{4^{n+1}}$. Radius of convergence is 2, since the nearest singularity (to 0) of $\dfrac{z}{z^2+4}$ is $\pm 2\mathbf{i}$, whose distance is 2 (that is, 2 is the largest number r such that $D(0;r)$ is still inside the domain of our function).

(c) $\dfrac{e^z}{(1-z)^2}$ centered at 0. (Radius of convergence is going to be 1, by the same argument as in (b)). Since $e^z = \displaystyle\sum_{k=0}^{\infty} \frac{z^k}{k!}$ and $\frac{1}{(1-z)^2} = \sum_{\ell=0}^{\infty} (\ell+1) z^{\ell}$ (by Problem 5 above), we can just multiply the two power series to get:

$$\frac{e^z}{(1-z)^2} = \sum_{n=0}^{\infty} \left(\sum_{\ell=0}^{n} \frac{\ell+1}{(n-\ell)!} \right) z^n.$$

Problem 11. For $f(z)$ and $\alpha \in \mathbb{C}$ given below, determine the nature of singularity of f at α: removable, pole or essential. If pole, determine its order.

(a) $f(z) = \dfrac{e^z - e^{-z}}{z}$, $\alpha = 0$. This is a removable singularity, since by l'hôpital rule: $\displaystyle\lim_{z \to 0} \frac{e^z - e^{-z}}{z} = 2$ exists.

(b) $f(z) = \dfrac{z-1}{z^5(z^2+9)}$, $\alpha = 0$. This is a pole of order 5, as $\displaystyle\lim_{z\to 0} z^5 f(z) = -\frac{1}{9} \neq 0$.

(c) $f(z) = e^{z - \frac{1}{z}}$, $\alpha = 0$. This is an essential singularity. The easiest way to prove it is by exclusion. If $f(z)$ had a removable singularity, or a pole of some order at 0, then we would be able to write $f(z) = z^{-n}\phi(z)$, where $n \in \mathbb{Z}_{\geq 0}$ and $\phi(z)$ is holomorphic near 0. But that would mean that $e^{-z^{-1}} = z^{-n}\phi(z)e^{-z}$ has a pole of order $\leq n$ at 0. This contradicts the fact that $e^{-z^{-1}}$ has an essential singularity at 0, from its Laurent series $e^{-z^{-1}} = \displaystyle\sum_{k=0}^{\infty} (-1)^k \frac{z^{-k}}{k!}$.

(d) $f(z) = \dfrac{1}{\cos(z)} = \sec(z)$. $\alpha = \frac{\pi}{2}$. It is a pole of order 1, since the limit (computed using l'hôpital rules again) $\lim_{z\to\frac{\pi}{2}} \left(z - \dfrac{\pi}{2}\right) f(z) = -1 \neq 0$.

(e) $f(z) = \dfrac{e^{z^2}-1}{z^4}$, $\alpha = 0$. This is a pole of order 2, since $\lim_{z\to 0} z^2 f(z) = \lim_{z\to 0} \dfrac{e^{z^2}-1}{z^2} = 1 \neq 0$.

(f) $f(z) = \dfrac{z}{\cos(z)-1}$, $\alpha = 2\pi$. It is a pole of order 2. We can compute the limit: $\lim_{z\to 2\pi} \dfrac{z(z-2\pi)^2}{\cos(z)-1} = -4\pi \neq 0$.

Problem 12. Consider the function $f(z) = \dfrac{1}{(z-1)(z-2)}$. Write its Taylor series expansion around 0. Write its Laurent series expansion near 1. Write its Taylor series expansion near ∞.

Solution. The Taylor series near 0 was already computed in Lecture 23, §23.5.
Laurent series near 1 can be obtained as follows:

$$\frac{1}{(z-1)(z-2)} = \frac{1}{z-1} \cdot \frac{-1}{1-(z-1)} = \frac{-1}{z-1} \sum_{k=0}^{\infty} (z-1)^k.$$

Finally, Taylor series near $z = \infty$ of $f(z)$ is nothing but Taylor series near $w = 0$ of $f(w)$, where $w = z^{-1}$.

$$\frac{1}{(z-1)(z-2)} = \frac{w^2}{(1-w)(1-2w)} = w^2 \left(\frac{2}{1-2w} - \frac{1}{1-w} \right) = \sum_{n=0}^{\infty} (2^{n+1}-1) w^{n+2} .$$

Hence, $\dfrac{1}{(z-1)(z-2)} = \sum_{n=0}^{\infty} \left(2^{n+1}-1\right) z^{-n-2}$ is the Taylor series expansion near ∞.

Problem 13. Prove that 0 is a removable singularity of $\dfrac{z}{e^z - 1}$. Consider its Taylor series expansion, centered at 0: $\dfrac{z}{e^z-1} = \sum_{n=0}^{\infty} b_n z^n$.

(1) Compute b_0, b_1, b_2 and b_3.

Solution. Since we have $z = (e^z - 1) \sum_{n=0}^{\infty} b_n z^n$, we can write:

$$z = \left(z + \frac{z^2}{2!} + \frac{z^3}{3!} + \frac{z^4}{4!} + \cdots \right) \cdot \left(b_0 + b_1 z + b_2 z^2 + b_3 z^3 + \cdots \right).$$

Comparing coefficients of z on both sides, we get $1 = b_0$. Comparing coefficients of z^2 on both sides, we get $0 = b_1 + \dfrac{b_0}{2} \Rightarrow b_1 = -\dfrac{1}{2}$.

For z^3, we obtain $0 = \dfrac{b_0}{6} + \dfrac{b_1}{2} + b_2 \Rightarrow b_2 = \dfrac{1}{12}$. Finally, for z^4, we get:

$$b_3 = -\frac{b_0}{24} - \frac{b_1}{6} - \frac{b_2}{2} = -\frac{1}{24} + \frac{1}{12} - \frac{1}{24} = 0.$$

(2) Prove that the radius of convergence of $\sum_{n=0}^{\infty} b_n z^n$ is 2π.

Solution. Since the singularities of $\dfrac{z}{e^z - 1}$ are at $\{2n\pi\mathbf{i} : n \in \mathbb{Z}_{\neq 0}\}$, the largest open disc we can have, centered at 0 which avoids this set is $D(0; 2\pi)$. Hence, the radius of convergence of the Taylor series is 2π.

(3) Prove that $b_{2k+1} = 0$ for every $k \geq 1$.

Solution. Let $f(z) = \dfrac{z}{e^z - 1}$. We compute $f(z) - f(-z)$ in two ways. First via the Taylor series near 0: $f(z) - f(-z) = 2\sum_{k=0}^{\infty} b_{2k+1} z^{2k+1}$. Next, using the definition,

$$f(z) - f(-z) = \frac{z}{e^z - 1} - \frac{-z}{e^{-z} - 1} = \frac{z}{e^z - 1} - \frac{ze^z}{e^z - 1} = -z.$$

Hence, $2b_1 = -1$ (which we already knew), and $b_{2k+1} = 0$ for every $k \geq 1$.

Problem 14. By multiplying the power series and using binomial formula [1], prove that: $e^z e^w = e^{z+w}$.

Solution. $e^z e^w = \left(\sum_{k=0}^{\infty} \dfrac{z^k}{k!}\right) \cdot \left(\sum_{\ell=0}^{\infty} \dfrac{w^\ell}{\ell!}\right)$. Multiplying the two power series, and collecting terms $z^a w^b$ with $a + b = n$, we get:

$$e^z e^w = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \frac{1}{k!} \frac{1}{(n-k)!} z^k w^{n-k}\right) = \sum_{n=0}^{\infty} \frac{1}{n!} \left(\sum_{k=0}^{n} \binom{n}{k} z^k w^{n-k}\right)$$
$$= \sum_{n=0}^{\infty} \frac{(z+w)^n}{n!} = e^{z+w}.$$

[1] Binomial formula: $(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$.