COMPLEX ANALYSIS: PROBLEM SHEET 6

Problem 1. Prove that $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges.

Problem 2. Find the radius of convergence of the following series.
(a) $\sum_{n=0}^{\infty} 7^n z^n$, (b) $\sum_{n=1}^{\infty} n^n z^n$, (c) $\sum_{n=1}^{\infty} \frac{z^n}{n^2}$.

(*For computing radius of convergence, read Lecture 22, §22.4 - ratio test.*)

Problem 3. Let $a, b, c \in \mathbb{C}$ be complex numbers such that $c \notin \mathbb{Z}_{\geq 0}$. The following power series is called *hypergeometric series.* [1].

$$F(a, b; c; z) = 1 + \sum_{n=1}^{\infty} \frac{a(a+1)\cdots(a+n-1)b(b+1)\cdots(b+n-1)}{c(c+1)\cdots(c+n-1)} \frac{z^n}{n!}.$$

Prove that: (i) If $a, b \in \mathbb{Z}_{\leq 0}$, then the radius of convergence of $F(a, b; c; z)$ is ∞. (ii) If $a, b \notin \mathbb{Z}_{\leq 0}$, then its radius of convergence is 1.

Problem 4. $f(z) = \ln(1-z)$ is an antiderivative of $\frac{-1}{1-z}$, with $f(0) = 0$. Use this fact, and the power series expansion (geometric series) $\frac{1}{1-z} = \sum_{k=0}^{\infty} z^k$ in the open disc $D(0; 1)$ to prove that

$$\ln(1-z) = -\sum_{n=1}^{\infty} \frac{z^n}{n}, \text{ for every } z \in D(0; 1).$$

Problem 5. For any $\ell \in \mathbb{Z}_{\geq 0}$, prove that:

$$\frac{1}{(1-z)^{\ell+1}} = \sum_{k=0}^{\infty} \binom{k+\ell}{\ell} z^k \text{ for every } z \in D(0; 1).$$

Recall $\binom{m}{n} = \frac{m!}{n!(m-n)!}$. (Hint: start from $\ell = 0$ and take derivatives (power series can be term–wise differentiated - Lecture 23, page 3).)

Problem 6. Let $\sum_{k=0}^{\infty} a_k z^k$ be a power series with radius of convergence $R > 0$.

[1] Carl Friedrich Gauss (1777-1855) studied this series in 1813. It was introduced by John Wallis (1616-1703) in his 1655 book *Arithmetica Infinitorum* who also coined the term hypergeometric. When $a = 1$ and $b = c$, $F(1, c; c; z)$ becomes the geometric series.

(1) Prove that the radius of convergence of $\displaystyle\sum_{k=0}^{\infty} a_k \frac{z^{k+1}}{k+1}$ is $\geq R$.

(2) Prove that the radius of convergence of $\displaystyle\sum_{k=0}^{\infty} a_k \frac{z^k}{k!}$ is ∞.

Problem 7. Find the mistake in the following calculation.

$$\frac{1}{1-z} = \sum_{k=0}^{\infty} z^k \qquad \text{and} \qquad \frac{1}{1-z} = -\frac{1}{z} \cdot \frac{1}{1-z^{-1}} = -\sum_{\ell=0}^{\infty} z^{-\ell-1}.$$

Taking the difference we get:

$$0 = \sum_{k=0}^{\infty} z^k + \sum_{\ell=0}^{\infty} z^{-\ell-1}.$$

Compare coefficient of (say) z to get $0 = 1$.

Problem 8. Let $\displaystyle\sum_{k=0}^{\infty} \frac{2^k}{k} z^{-k}$ be a power series centered at ∞. What is its radius of convergence? (see Lecture 23, §23.6).

Problem 9. Compute the Taylor series expansion of $\cos(z)$, around 0, in the following two ways:

(1) By computing $\displaystyle\frac{d^n}{dz^n}(\cos(z))$ at $z = 0$.

(2) Using $\cos(z) = \displaystyle\frac{e^{\mathbf{i}z} + e^{-\mathbf{i}z}}{2}$ and the Taylor series of e^z.

Problem 10. Compute the Taylor series expansion, and determine its radius of convergence:

(a) e^z centered at 1, (b) $\displaystyle\frac{z}{z^2+4}$ centered at 0, (c) $\displaystyle\frac{e^z}{(1-z)^2}$ centered at 0.

Problem 11. For $f(z)$ and $\alpha \in \mathbb{C}$ given below, determine the nature of singularity of f at α: removable, pole or essential (see Lecture 24, §24.5, 24.6). If pole, determine its order.

(a) $f(z) = \displaystyle\frac{e^z - e^{-z}}{z}$, $\alpha = 0$, (b) $f(z) = \displaystyle\frac{z-1}{z^5(z^2+9)}$, $\alpha = 0$,

(c) $f(z) = e^{z-\frac{1}{z}}$, $\alpha = 0$, (d) $f(z) = \displaystyle\frac{1}{\cos(z)} = \sec(z)$. $\alpha = \frac{\pi}{2}$,

(e) $f(z) = \displaystyle\frac{e^{z^2} - 1}{z^4}$, $\alpha = 0$, (f) $f(z) = \displaystyle\frac{z}{\cos(z) - 1}$, $\alpha = 2\pi$.

Problem 12. Consider the function $f(z) = \dfrac{1}{(z-1)(z-2)}$. Write its Taylor series expansion around 0. Write its Laurent series expansion near 1. Write its Taylor series expansion near ∞.

Problem 13. Prove that 0 is a removable singularity of $\dfrac{z}{e^z - 1}$. Consider its Taylor series expansion, centered at 0:

$$\frac{z}{e^z - 1} = \sum_{n=0}^{\infty} b_n z^n \ .$$

(1) Compute b_0, b_1, b_2 and b_3.

(2) Prove that the radius of convergence of $\sum_{n=0}^{\infty} b_n z^n$ is 2π.

(3) Prove that $b_{2k+1} = 0$ for every $k \geq 1$.
(*Hint: if* $f(z) = \dfrac{z}{e^z - 1}$, *what is* $f(z) - f(-z)$*?*)

Problem 14. By multiplying the power series and using binomial formula [2], prove that: $e^z e^w = e^{z+w}$.

[2]Binomial formula: $(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$.