PROBLEM SHEET 7: SOLUTIONS

Problem 1. Compute the following residues.

(a) $\operatorname*{Res}_{z=2\pi\mathbf{i}}\left(\dfrac{1}{z(1-e^{-z})}\right)$.

Solution. Change of variables $z = x + 2\pi\mathbf{i}$ changes it to (since $e^{-x-2\pi\mathbf{i}} = e^{-x}$):

$$\operatorname*{Res}_{x=0}\left(\frac{1}{(x+2\pi\mathbf{i})(1-e^{-x})}\right) = \operatorname*{Res}_{x=0}\left(\frac{1}{x}\boxed{\frac{1}{x+2\pi\mathbf{i}}\cdot\frac{x}{1-e^{-x}}}\right)$$

The term in the box is defined at $x = 0$. So the pole at $x = 0$ is of order 1 and residue there is the value of the function in the box, at $x = 0$, namely: $\dfrac{1}{2\pi\mathbf{i}}$ (since $\lim\limits_{x\to 0}\dfrac{x}{1-e^{-x}} = 1$.)

(b) $\operatorname*{Res}_{z=3\pi}(\cot(z))$.

Solution. Change of variables $z = x + 3\pi\mathbf{i}$ changes it to ($\cos(x+3\pi) = -\cos(x)$ and $\sin(x+3\pi) = -\sin(x)$, so $\cot(x+3\pi) = \cot(x)$):

$$\operatorname*{Res}_{x=0}\left(\frac{\cos(x)}{\sin(x)}\right) = \operatorname*{Res}_{x=0}\left(\frac{1}{x}\boxed{\cos(x)\cdot\frac{x}{\sin(x)}}\right)$$

So, the pole is of order 1, and residue there is the value of the function in the box at $x = 0$, namely 1, since $\cos(0) = 1$ and $\lim\limits_{x\to 0}\dfrac{x}{\sin(x)} = 1$.

(c) $\operatorname*{Res}_{z=0}\left(\dfrac{1}{z^2\sin(z)}\right)$.

Solution. Multiply and divide the function by z to write:

$$\operatorname*{Res}_{z=0}\left(\frac{1}{z^2\sin(z)}\right) = \operatorname*{Res}_{z=0}\left(\frac{1}{z^3}\boxed{\frac{z}{\sin(z)}}\right)$$

So, the pole at $z = 0$ is of order 3 and residue there is computed by either $\dfrac{1}{2}\left[\dfrac{d^2}{dz^2}\dfrac{z}{\sin z}\right]_{z=0}$, or by computing the coefficient of z^2 in $\dfrac{z}{\sin(z)}$. I prefer the latter, and the computation is:

$$\frac{z}{\sin(z)} = a_0 + a_1 z + a_2 z^2 + \cdots \;\Rightarrow\; z = \left(z - \frac{z^3}{6} + \cdots\right)(a_0 + a_1 z + a_2 z^2 + \cdots)$$

Comparing coefficients of z gives $1 = a_0$; of z^2 gives $0 = a_1$; of z^3 gives $0 = a_2 - \frac{a_0}{6} \Rightarrow$ $a_2 = \frac{1}{6}$. Hence $\operatorname*{Res}_{z=0}\left(\frac{1}{z^2\sin(z)}\right) = \frac{1}{6}$.

(d) $\operatorname*{Res}_{z=n\pi}\left(\frac{1}{z^2\sin(z)}\right)$, $(n \in \mathbb{Z}_{\neq 0})$.

Solution. Change variables $z = x + n\pi$ and use $\sin(x + n\pi) = (-1)^n$ to rewrite the problem as:

$$\operatorname*{Res}_{x=0}\left(\frac{(-1)^n}{(x+n\pi)^2\sin(x)}\right) = \operatorname*{Res}_{x=0}\left(\frac{1}{x}\boxed{\frac{(-1)^n}{(x+n\pi)^2}\cdot\frac{x}{\sin(x)}}\right)$$

Again, using $\lim_{x\to 0}\frac{x}{\sin(x)} = 1$, the value of the function in the box, at $x = 0$ is $\frac{(-1)^n}{n^2\pi^2}$.

(e) $\operatorname*{Res}_{z=0}\left(\frac{z-\sin(z)}{z}\right)$.

Solution. Since $z - \sin(z) = \frac{z^3}{6} - \cdots$, dividing it by z still gives a function defined at 0. Hence the residue is 0.

(f) $\operatorname*{Res}_{z=0}\left(\frac{e^z - e^{-z}}{z^4(1-z^2)}\right)$.

Solution. The answer is the coefficient of z^3 in $\frac{e^z-e^{-z}}{1-z^2}$, which we can compute as follows:

$$\frac{e^z-e^{-z}}{1-z^2} = (e^z - e^{-z})\frac{1}{1-z^2} = 2\left(z + \frac{z^3}{3!} + \cdots\right)\left(1 + z^2 + z^4 + \cdots\right)$$

So, the coefficient of z^3 is $2\left(1 + \frac{1}{6}\right) = \frac{7}{3}$.

(g) $\operatorname*{Res}_{z=0}\left(\frac{\ln(1+z)\sin(z)}{z^5}\right)$.

Solution. Again, the answer is the coefficient of z^4 in $\ln(1+z)\sin(z)$ which we compute as follows:

$$\ln(1+z)\sin(z) = \left(z - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + \cdots\right)\left(z - \frac{z^3}{6} + \frac{z^5}{5!} - \cdots\right)$$

Coefficient of $z^4 = -\frac{1}{6} + \frac{1}{3} = \frac{1}{6}$.

Problem 2. Prove that $\operatorname*{Res}_{z=\infty}\left(\frac{2z^3+7}{z(z-1)^3}\right) = -2$.

Solution. (see Lecture 26, §26.2) Use $\operatorname*{Res}_{z=\infty}(f(z)) = -\operatorname*{Res}_{w=0}\left(w^{-2}f(w)\right)$ where $w = z^{-1}$.

$$\operatorname*{Res}_{z=\infty}\left(\frac{2z^3+7}{z(z-1)^3}\right) = -\operatorname*{Res}_{w=0}\left(\frac{1}{w^2}\cdot\frac{2+7w^3}{w^3}\cdot\frac{w^4}{(1-w)^3}\right) = -\operatorname*{Res}_{w=0}\left(\frac{1}{w}\boxed{\frac{2+7w^3}{(1-w)^3}}\right)$$

which is equal to (-1) times the value of the function in the box at $w=0$, that is -2.

Problem 3. Let C be the counterclockwise circle of radius 3, centered at 0. Compute the following integral, using the change of variables $w = z^{-1}$: $\displaystyle\int_C \frac{z^3 e^{\frac{1}{z}}}{1+z^3}\,dz$.

Solution. Change of variables $w = z^{-1}$ changes C into a clockwise circle (denoted by $-C'$), centered at 0, of radius $1/3$; dz into $-w^{-2}dw$. Hence:

$$\int_C \frac{z^3 e^{\frac{1}{z}}}{1+z^3}\,dz = \int_{-C'} \frac{w^{-3}e^w}{1+w^{-3}}\,(-w^{-2}dw) = \int_{C'} \frac{e^w}{w^3+1}\frac{dw}{w^2}$$

By Cauchy's integral formula, this integral is given by: $2\pi\mathbf{i}\left[\dfrac{d}{dw}\dfrac{e^w}{w^3+1}\right]_{w=0}$, which is equal to $2\pi\mathbf{i}\left[\dfrac{(w^3+1)e^w - e^w\cdot 3w^2}{(w^3+1)^2}\right]_{w=0} = 2\pi\mathbf{i}$.

Problem 4. Let $n < m$ be two positive integers. Let $P(z) = a_n z^n + a_{n-1}z^{n-1} + \cdots + a_0$ and $Q(z) = b_m z^m + \cdots + b_0$ be two polynomials of degrees n and m respectively. Prove that $\operatorname*{Res}_{z=\infty}\left(\dfrac{P(z)}{Q(z)}\right) = \begin{cases} 0 & \text{if } n < m-1 \\ -\frac{a_n}{b_m} & \text{if } n = m-1 \end{cases}$.

Solution. I am again going to use $\operatorname*{Res}_{z=\infty}(f(z)) = -\operatorname*{Res}_{w=0}\left(w^{-2}f(w)\right)$ where $w = z^{-1}$.

$$\begin{aligned}\operatorname*{Res}_{z=\infty}\left(\frac{P(z)}{Q(z)}\right) &= -\operatorname*{Res}_{w=0}\left(w^{-2}\frac{P(w^{-1})}{Q(w^{-1})}\right)\\ &= -\operatorname*{Res}_{w=0}\left(\frac{w^m}{w^{n+2}}\boxed{\frac{a_n + a_{n-1}w + \cdots + a_0 w^n}{b_m + b_{m-1}w + \cdots + b_0 w^m}}\right)\end{aligned}$$

Now $m \geq n+1$ is given. If $m \geq n+2$, the function written above is holomorphic at $w=0$, so its residue at $w=0$ is 0. If $m = n+1$, then the residue is simply (-1) times the value of the fraction in the box at $w=0$ equal to $-\dfrac{a_n}{b_m}$.

Problem 5. Let $\Omega \subset \mathbb{C}$ be an open set and $f : \Omega \to \mathbb{C}$ be a holomorphic function. Assume that $z_0 \in \Omega$ is a zero of f of order $N \in \mathbb{Z}_{\geq 1}$. Prove that $\operatorname*{Res}_{z=z_0}\left(\dfrac{f'(z)}{f(z)}\right) = N$.

Solution. f vanishes at z_0 to order N means $f(z) = (z-z_0)^N g(z)$, and g does not vanish at all in a small enough disc around z_0. This allows us to compute:

$$\frac{f'(z)}{f(z)} = \frac{N(z-z_0)^{N-1}g(z) + (z-z_0)^N g'(z)}{(z-z_0)^N g(z)} = \frac{N}{z-z_0} + \frac{g'(z)}{g(z)}.$$

Now $\dfrac{g'(z)}{g(z)}$ is holomorphic near z_0, hence has residue 0 at z_0. So,

$$\operatorname*{Res}_{z=z_0}\left(\frac{f'(z)}{f(z)}\right) = \operatorname*{Res}_{z=z_0}\left(\frac{N}{z-z_0}\right) = N.$$

Problem 6. Let $\Omega \subset \mathbb{C}$ be an open set, $\alpha \in \Omega$, and $f : \Omega \setminus \{\alpha\} \to \mathbb{C}$ be a holomorphic function, such that α is a pole of f, of order $M \in \mathbb{Z}_{\geq 1}$. Prove that $\operatorname*{Res}_{z=\alpha}\left(\frac{f'(z)}{f(z)}\right) = -M$.

Solution. Same as Problem 5, just change N to $-M$.

Problem 7. Compute the following integrals:

(a) $\displaystyle\int_0^{2\pi} \frac{d\theta}{5+4\sin(\theta)}$.

Solution. Write $z = e^{\mathbf{i}\theta}$ so that $\sin(\theta) = \dfrac{z-z^{-1}}{2\mathbf{i}}$ and $d\theta = \dfrac{dz}{\mathbf{i}z}$. Let C be the counterclockwise circle around 0 of radius 1. Then the integral we have to compute is:

$$\int_C \frac{1}{5+4\left(\frac{z-z^{-1}}{2\mathbf{i}}\right)} \frac{dz}{\mathbf{i}z} = \int_C \frac{dz}{2z^2+5\mathbf{i}z-2}\ .$$

Now the quadratic equation $2z^2+5\mathbf{i}z-2=0$ has two solutions: $\alpha_1 = -\dfrac{\mathbf{i}}{2}$ and $\alpha_2 = -2\mathbf{i}$. α_1 is within C and α_2 is outside. So, we can compute the integral using Cauchy's formula:

$$\int_C \frac{dz}{2z^2+5\mathbf{i}z-2} = \int_C \frac{dz}{2(z+2\mathbf{i})\left(z+\frac{\mathbf{i}}{2}\right)} = 2\pi\mathbf{i}\frac{1}{2\left(2\mathbf{i}-\frac{\mathbf{i}}{2}\right)}$$

which gives the answer: $\dfrac{2\pi}{3}$.

(b) $\displaystyle\int_0^{2\pi} \frac{d\theta}{(a+\cos(\theta))^2}$, $(a \in \mathbb{R}_{>1})$.

Solution. Again, change of variables $z = e^{\mathbf{i}\theta}$ turns the integral in question to:

$$\int_C \frac{1}{\left(a+\frac{z+z^{-1}}{2}\right)^2}\frac{dz}{\mathbf{i}z} = \frac{4}{\mathbf{i}}\int_C \frac{z}{(z^2+2az+1)^2}\,dz.$$

$z^2+2az+1=0$ has two solutions: $\alpha_1 = -a+\sqrt{a^2-1}$ and $\alpha_2 = -a-\sqrt{a^2-1}$. As $a>1$, $|\alpha_2| = a+\sqrt{a^2-1} > 1$, hence α_2 is outside of C. As $\alpha_1\alpha_2 = 1$, α_1 must be inside C.

$$\frac{4}{\mathbf{i}}\int_C \frac{z}{(z^2+2az+1)^2}\,dz = \frac{4}{\mathbf{i}}\int_C \frac{z}{(z-\alpha_1)^2(z-\alpha_2)^2}\,dz = \frac{4}{\mathbf{i}}2\pi\mathbf{i}\left[\frac{d}{dz}\left(\frac{z}{(z-\alpha_2)^2}\right)\right]_{z=\alpha_1}$$

$$= 8\pi\left[\frac{1}{(z-\alpha_2)^2} - \frac{2z}{(z-\alpha_2)^3}\right]_{z=\alpha_1} = 8\pi\frac{-(\alpha_1+\alpha_2)}{(\alpha_1-\alpha_2)^3}$$

$$= 8\pi\frac{2a}{(2\sqrt{a^2-1})^3} = \frac{2\pi a}{(a^2-1)^{\frac{3}{2}}}.$$

(c) $\displaystyle\int_0^{2\pi} \frac{d\theta}{1+\sin^2(\theta)}$.

Solution. $z = e^{\mathbf{i}\theta}$ changes the problem to:

$$\int_C \frac{1}{1+\left(\frac{z-z^{-1}}{2\mathbf{i}}\right)^2}\,\frac{dz}{\mathbf{i}z} = -\frac{4}{\mathbf{i}}\int_C \frac{z}{z^4-6z^2+1}\,dz.$$

Now, $z^4-6z^2+1=0$ is a quadratic equation in z^2, which gives us two solutions $z^2 = r_1, r_2$, where:

$$r_1 = 3+2\sqrt{2} \qquad \text{and} \qquad r_2 = 3-2\sqrt{2}.$$

r_1 is clearly outside of C; and because of $r_1r_2=1$, we must have r_1 inside C. Thus, $z^4-6z^2+1 = (z^2-r_1)(z^2-r_2) = (z^2-r_1)(z-\sqrt{r_2})(z+\sqrt{r_2})$. Our integral can now again be computed using Cauchy's formula:

$$\begin{aligned}
-\frac{4}{\mathbf{i}}\int_C \frac{z}{z^4-6z^2+1}\,dz &= -\frac{4}{\mathbf{i}}\int_C \frac{z}{(z^2-r_1)(z-\sqrt{r_2})(z+\sqrt{r_2})}\,dz \\
&= -\frac{4}{\mathbf{i}}2\pi\mathbf{i}\left(\frac{\sqrt{r_2}}{(r_2-r_1)2\sqrt{r_1}} + \frac{-\sqrt{r_2}}{(r_2-r_1)(-2\sqrt{r_2})}\right) \\
&= \frac{8\pi}{r_1-r_2} = \frac{8\pi}{4\sqrt{2}} = \sqrt{2}\pi.
\end{aligned}$$

(d) $\displaystyle\int_0^{2\pi} \frac{\cos(2\theta)\ d\theta}{1-2p\cos(\theta)+p^2}$, $(0<p<1)$.

Solution. Perform the change of variables $z = e^{\mathbf{i}\theta}$ again to convert the integral in question to:

$$\int_C \frac{\frac{z+z^{-1}}{2}}{1-2p\left(\frac{z+z^{-1}}{2}\right)+p^2}\,\frac{dz}{\mathbf{i}z} = \frac{1}{2\mathbf{i}}\int_C \frac{z^4+1}{z^2(z-pz^2-p+p^2z)}\,dz.$$

Using the factorization: $z-pz^2-p+p^2z = (z-p)(1-pz)$, we see that the integrand has 2 singularities inside C: at $z=0$ and $z=p$ (since $0<p<1$). Let C_1 be a small (counterclockwise) circle around p and C_2 around 0.

$$\frac{1}{2\mathbf{i}}\int_C \frac{z^4+1}{z^2(z-p)(1-pz)}\,dz = \frac{1}{2\mathbf{i}}\int_{C_1} \frac{z^4+1}{z^2(z-p)(1-pz)}\,dz + \frac{1}{2\mathbf{i}}\int_{C_2} \frac{z^4+1}{z^2(z-p)(1-pz)}\,dz$$

Now,

$$\begin{aligned}
\frac{1}{2\mathbf{i}}\int_{C_1} \frac{z^4+1}{z^2(z-p)(1-pz)}\,dz &= \frac{2\pi\mathbf{i}}{2\mathbf{i}}\cdot\frac{p^4+1}{p^2(1-p^2)} = \pi\frac{p^4+1}{p^2(1-p^2)}. \\
\frac{1}{2\mathbf{i}}\int_{C_2} \frac{z^4+1}{z^2(z-p)(1-pz)}\,dz &= \frac{2\pi\mathbf{i}}{2\mathbf{i}}\cdot\left[\frac{d}{dz}\left(\frac{z^4+1}{(z-p)(1-pz)}\right)\right]_{z=0} \\
&= \pi\left[\frac{4z^3}{(z-p)(1-pz)} - \frac{z^4+1}{(z-p)^2(1-pz)} + \frac{p(z^4+1)}{(z-p)(1-pz)^2}\right]_{z=0} \\
&= -\pi\frac{1+p^2}{p^2}.
\end{aligned}$$

Combining, we get:

$$\frac{1}{2\mathbf{i}}\int_C \frac{z^4+1}{z^2(z-p)(1-pz)}\,dz = \pi\left(\frac{1+p^4}{p^2(1-p^2)} - \frac{1+p^2}{p^2}\right)$$

$$= \pi\frac{1+p^4-(1-p^2)(1+p^2)}{p^2(1-p^2)} = \pi\frac{1+p^4-(1-p^4)}{p^2(1-p^2)} = \frac{2\pi p^2}{1-p^2}.$$

Problem 8. For $n \in \mathbb{Z}_{\geq 0}$, prove that $\displaystyle\int_0^{\pi} \sin^{2n}(\theta)\,d\theta = \frac{(2n)!}{2^{2n}(n!)^2}\pi$.

Solution. Let us compute $\displaystyle\int_0^{2\pi} \sin^{2n}(\theta)\,d\theta$ first (we will divide it by 2 in the end). Again we do the change of variables $z = e^{\mathbf{i}\theta}$. Our function is going to become:

$$\sin^{2n}(\theta) = \left(\frac{z-z^{-1}}{2\mathbf{i}}\right)^{2n} = \frac{(-1)^n}{2^{2n}}(z-z^{-1})^{2n}$$

And the integral we are trying to compute is:

$$\int_0^{2\pi} \sin^{2n}(\theta)\,d\theta = \int_C \frac{(-1)^n}{2^{2n}}(z-z^{-1})^{2n}\frac{dz}{\mathbf{i}z} = \frac{(-1)^n}{2^{2n}\mathbf{i}}\int_C (z-z^{-1})^{2n}\frac{dz}{z}\ .$$

We can expand $(z-z^{-1})^{2n}$ using the binomial formula:

$$(z-z^{-1})^{2n} = \sum_{k=0}^{2n}(-1)^k\binom{2n}{k}z^{2n-k}(z^{-1})^k = \sum_{k=0}^{2n}(-1)^k\binom{2n}{k}z^{2n-2k}$$

Therefore our integral becomes:

$$\frac{(-1)^n}{2^{2n}\mathbf{i}}\int_C (z-z^{-1})^{2n}\frac{dz}{z} = \frac{(-1)^n}{2^{2n}\mathbf{i}}\cdot\left(\sum_{k=0}^{2n}(-1)^k\binom{2n}{k}\int_C z^{2n-2k-1}\,dz\right)$$

Using the fact that $\int_C z^{\ell}\,dz$ is $2\pi\mathbf{i}$ for $\ell = -1$ and 0 otherwise, we see that the only term in the sum that gives a non–zero integral is when $k = n$. So, our integral simplifies to:

$$\int_0^{2\pi} \sin^{2n}(\theta)\,d\theta = \frac{(-1)^n}{2^{2n}\mathbf{i}}2\pi\mathbf{i}(-1)^n\binom{2n}{n} = \frac{2\pi}{2^{2n}}\frac{(2n)!}{(n!)^2}.$$

Hence, $\displaystyle\int_0^{\pi} \sin^{2n}(\theta)\,d\theta = \frac{\pi}{2^{2n}}\frac{(2n)!}{(n!)^2}$.

Problem 9. Compute the following integrals.

(a) $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{x^4+1}$.

Solution. We are going to integrate $\dfrac{1}{z^4+1}$ along C_R. See the figure below.

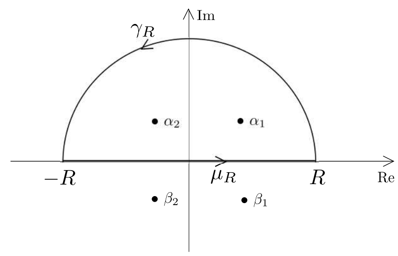

FIGURE 1. Contour C_R consisting of two smooth pieces.

Step 1. Let us estimate the function $\frac{1}{z^4+1}$ as z lies on γ_R. By triangle inequality $|z^4+1| \geq |z^4| - 1 = R^4 - 1$, so by our *important* inequality:

$$\left| \int_{\gamma_R} \frac{dz}{z^4+1} \right| \leq \frac{1}{R^4-1} \cdot \pi R \to 0 \text{ as } R \to \infty.$$

Step 2. $\int_{C_R} \frac{1}{z^4+1}\, dz$. (This computation is very similar to the one given in Lecture 27, §27.5). The solutions of $z^4 = -1$ are the following (see Figure above).

$$\alpha_1 = e^{\frac{\pi}{4}\mathbf{i}}, \quad \alpha_2 = e^{\frac{3\pi}{4}\mathbf{i}}$$

$$\beta_2 = e^{\frac{5\pi}{4}\mathbf{i}}, \quad \beta_1 = e^{\frac{7\pi}{4}\mathbf{i}}$$

So, $\int_{C_R} \frac{1}{z^4+1}\, dz = 2\pi\mathbf{i} \left(\operatorname*{Res}_{z=\alpha_1} \left(\frac{1}{z^4+1} \right) + \operatorname*{Res}_{z=\alpha_2} \left(\frac{1}{z^4+1} \right) \right)$.

$$\operatorname*{Res}_{z=\alpha_1} \left(\frac{1}{z^4+1} \right) = \lim_{z\to\alpha_1} \frac{z-\alpha_1}{z^4+1} = \lim_{z\to\alpha_1} \frac{1}{4z^3} = \frac{1}{4\alpha_1^3}$$

$$= \frac{\alpha_1}{4\alpha_1^4} = -\frac{\alpha_1}{4}$$

Now, $\alpha_1 + \alpha_2 = e^{\mathbf{i}\pi/4} + e^{\mathbf{i}3\pi/4} = e^{\mathbf{i}\pi/4} - e^{-\mathbf{i}\pi/4} = 2\mathbf{i}\sin(\pi/4) = \sqrt{2}\mathbf{i}$.

Thus, the sum of the residues is:

$$\int_{C_R} \frac{1}{z^4+1}\, dz = -2\pi\mathbf{i}\frac{\alpha_1+\alpha_2}{4} = -2\pi\mathbf{i}\frac{\sqrt{2}\mathbf{i}}{4} = \frac{\pi}{\sqrt{2}}.$$

Final step. The integral in question $\int_{-\infty}^{\infty} \frac{dx}{x^4+1}$ is the limit

$$\lim_{R\to\infty} \left(\int_{C_R} \frac{1}{z^4+1}\, dz - \int_{\gamma_R} \frac{1}{z^4+1}\, dz \right).$$

The latter being zero, we get that the answer is $\frac{\pi}{\sqrt{2}}$.

(b) $\displaystyle\int_{-\infty}^{\infty} \frac{x^2}{(x^2+9)(x^2+4)^2}\, dx.$

Solution. We again are going to compute the integral over C_R (contour from the previous problem). The singularities of the integrand are $\pm 2\mathbf{i}$ and $\pm 3\mathbf{i}$, of which $2\mathbf{i}, 3\mathbf{i}$ are within C_R and the other two are outside.

Step 1. Figure out the bound of the integrand as z lies on γ_R:

$$\left| \int_{\gamma_R} \frac{z^2}{(z^2+9)(z^2+4)^2}\, dz \right| \leq \frac{R^2}{(R^2-9)(R^2-4)^2} \cdot \pi R \to 0 \text{ as } R \to \infty.$$

Step 2. Compute the integral over C_R:

$$\int_{C_R} \frac{z^2}{(z^2+9)(z^2+4)^2}\, dz = \int_{C_R} \frac{z^2}{(z+3\mathbf{i})(z+2\mathbf{i})^2} \frac{dz}{(z-3\mathbf{i})(z-2\mathbf{i})^2}$$

$$2\pi\mathbf{i} \left(\left[\frac{z^2}{(z+3\mathbf{i})(z^2+4)^2} \right]_{z=3\mathbf{i}} + \left[\frac{d}{dz} \left(\frac{z^2}{(z^2+9)(z+2\mathbf{i})^2} \right) \right]_{z=2\mathbf{i}} \right)$$

$$= 2\pi\mathbf{i} \left(\frac{(3\mathbf{i})^2}{(-9+4)^2(6\mathbf{i})} + \left[\frac{2z}{(z^2+9)(z+2\mathbf{i})^2} - \frac{z^2.2z}{(z^2+9)^2(z+2\mathbf{i})^2} - \frac{2z^2}{(z^2+9)(z+2\mathbf{i})^3} \right]_{z=2\mathbf{i}} \right)$$

$$= 2\pi\mathbf{i} \left(\frac{-9}{25(6\mathbf{i})} + \frac{4\mathbf{i}}{5(4\mathbf{i})^2} - \frac{(2\mathbf{i})^2 4\mathbf{i}}{(25)(4\mathbf{i})^2} - \frac{2(2\mathbf{i})^2}{5(4\mathbf{i})^3} \right)$$

$$= 2\pi \left(\frac{-3}{50} + \frac{1}{20} + \frac{1}{25} - \frac{1}{40} \right) = \frac{\pi}{100}.$$

Final step. Put everything together to get:

$$\int_{-\infty}^{\infty} \frac{x^2}{(x^2+9)(x^2+4)^2}\, dx = \frac{\pi}{100}.$$

Problem 10. Let $a, b \in \mathbb{R}_{>0}$ be two positive real numbers. Prove that:

$$\int_0^{\infty} \frac{x^4}{(a+bx^2)^4}\, dx = \frac{\pi}{32 a^{\frac{3}{2}} b^{\frac{5}{2}}}.$$

Solution. Let us write $t = a/b$ so that the integrand has singularities at $\pm t\mathbf{i}$. Again, we are going to integrate $\dfrac{z^4}{(a+bz^2)^4}$ over the contour C_R from the figure above (we will have to divide the answer by 2 at the end, since the problem is only asking for $\displaystyle\int_0^{\infty}$.)

Step 1. This is pretty much the same as the ones given before. We estimate the integral:

$$\left| \int_{\gamma_R} \frac{z^4}{b^4(z^2+t)^4}\, dz \right| \leq \frac{R^4}{b^4(R^2-t)^4} \cdot \pi R \to 0 \text{ as } R \to \infty.$$

Step 2. This is going to be a lengthy computation.

$$\frac{1}{b^4} \int_{C_R} \frac{z^4}{(z+t\mathbf{i})^4(z-t\mathbf{i})^4}\, dz = \frac{2\pi\mathbf{i}}{b^4} \cdot X\ ,$$

where X can be computed either as $\frac{1}{6}\left[\frac{d^3}{dz^3}\left(\frac{z^4}{(z+t\mathbf{i})^4}\right)\right]_{z=t\mathbf{i}}$, or as the coefficient of $(z-t\mathbf{i})^3$ in the Taylor series expansion of $\frac{z^4}{(z+t\mathbf{i})^4}$ near $z=t\mathbf{i}$. I like the second way (after computing first derivative it becomes clear that the calculation is only going to get messy).

So, a little change of variable $z = w + t\mathbf{i}$ turns our problem to:

$$X = \text{ Coefficient of } w^3 \text{ in the Taylor series of } \left(\frac{w+t\mathbf{i}}{w+2t\mathbf{i}}\right)^4 \text{ near } w=0.$$

$$\left(\frac{w+t\mathbf{i}}{w+2t\mathbf{i}}\right)^4 = (w+t\mathbf{i})^4 \cdot \frac{1}{(2t\mathbf{i})^4} \cdot \frac{1}{(1+\frac{w}{2t\mathbf{i}})^4}$$

$$= \frac{1}{(2t\mathbf{i})^4}(w^4+4w^3(t\mathbf{i})+6w^2(t\mathbf{i})^2+4w(t\mathbf{i})^3+(t\mathbf{i})^4)\left(1-4\frac{w}{2t\mathbf{i}}+10\frac{w^2}{(2t\mathbf{i})^2}-20\frac{w^3}{(2t\mathbf{i})^3}+\cdots\right)$$

(here I opened the first term using binomial formula, and the second using $\frac{1}{(1-z)^{\ell+1}} = \sum_{n=0}^{\infty}\begin{pmatrix} n+\ell \\ \ell \end{pmatrix} z^n$.)

So, the coefficient of w^3 in this product is:

$$X = \frac{1}{16t^4}(t\mathbf{i})\left(4-12+10-\frac{5}{2}\right) = \frac{\mathbf{i}}{16t^3}\left(-\frac{1}{2}\right).$$

Hence, we get:

$$\frac{1}{b^4}\int_{C_R}\frac{z^4}{(z+t\mathbf{i})^4(z-t\mathbf{i})^4}\,dz = \frac{2\pi\mathbf{i}}{b^4}\frac{\mathbf{i}}{16t^3}\left(\frac{-1}{2}\right) = \frac{\pi}{16t^3b^4} = \frac{\pi}{16a^{\frac{3}{2}}b^{4-\frac{3}{2}}}.$$

(since $t = ab^{-1}$.)

Final step. Gathering the results from the previous two steps, and dividing by 2, we get:

$$\int_0^\infty \frac{x^4}{(a+bx^2)^4}\,dx = \frac{\pi}{32a^{\frac{3}{2}}b^{\frac{5}{2}}}.$$

Problem 11. (Bonus) Let $n \in \mathbb{Z}_{\geq 0}$. Prove that

$$\int_0^{2\pi} e^{\cos(\theta)}\cos(n\theta-\sin(\theta))\,d\theta = \frac{2\pi}{n!} \qquad \text{and} \qquad \int_0^{2\pi} e^{\cos(\theta)}\sin(n\theta-\sin(\theta))\,d\theta = 0.$$

Solution. We are going to combine the two integrals as real and imaginary parts of one:

$$\cos(n\theta-\sin(\theta))+\mathbf{i}\sin(n\theta-\sin(\theta)) = e^{\mathbf{i}(n\theta-\sin(\theta))} = e^{\mathbf{i}n\theta}e^{-\mathbf{i}\sin(\theta)}.$$

Meaning, let $A = \displaystyle\int_0^{2\pi} e^{\cos(\theta)} \cos(n\theta - \sin(\theta))\, d\theta$ and $B = \displaystyle\int_0^{2\pi} e^{\cos(\theta)} \sin(n\theta - \sin(\theta))\, d\theta$, so that:

$$A + \mathbf{i}B = \int_0^{2\pi} e^{\cos(\theta)} e^{\mathbf{i}(n\theta - \sin(\theta))}\, d\theta = \int_0^{2\pi} e^{\cos(\theta) - \mathbf{i}\sin(\theta)} e^{-\mathbf{i}n\theta}\, d\theta.$$

Now do the substitution: $z = e^{\mathbf{i}\theta}$. Again, let C be the counterclockwise oriented circle of radius 1, centered at 0. Our integral turns into:

$$\int_C e^{z^{-1}} z^n \frac{dz}{\mathbf{i}z} = \frac{1}{\mathbf{i}} \int_C e^{z^{-1}} z^{n-1}\, dz = 2\pi \operatorname*{Res}_{z=0} \left(z^{n-1} e^{z^{-1}} \right).$$

Now, we have to use the series expansion of $e^{z^{-1}}$. We are looking for the coefficient of z^{-1} in the product $z^{n-1}e^{z^{-1}}$:

$$z^{n-1}e^{z^{-1}} = \sum_{k=0}^{\infty} \frac{z^{n-1-k}}{k!}$$

So, the coefficient of z^{-1} is $\dfrac{1}{n!}$. Hence,

$$A + B\mathbf{i} = 2\pi \operatorname*{Res}_{z=0} \left(z^{n-1} e^{z^{-1}} \right) = \frac{2\pi}{n!} \;\Rightarrow\; A = \frac{2\pi}{n!} \text{ and } B = 0.$$

Problem 12. Let $\Omega \subset \mathbb{C}$ be an open set, and let $f : \Omega \dashrightarrow \mathbb{C}$ be a meromorphic function. (Recall: this means that there is a subset $A \subset \Omega$, such that f is defined and holomorphic on $\Omega \setminus A$; and every point of A is a pole of f).
Let $\gamma : [a, b] \to \Omega$ be a counterclockwise oriented contour, which does not pass through any of the poles of f.

- Let $z_1, z_2, \dots, z_k \in \text{Interior}(\gamma)$ be zeroes of $f(z)$ which are inside γ, of orders $N_1, N_2, \dots, N_k$ respectively.

- Let $\alpha_1, \alpha_2, \dots, \alpha_\ell \in \text{Interior}(\gamma)$ be poles of f which are inside γ, of orders $M_1, M_2, \dots, M_\ell$ respectively.

Prove that $\displaystyle\frac{1}{2\pi\mathbf{i}} \int_\gamma \frac{f'(z)}{f(z)}\, dz = \sum_{a=1}^{k} N_a - \sum_{b=1}^{\ell} M_b.$

Solution. This is nothing but Problems 5 and 6, combined with Cauchy's residue theorem.