COMPLEX ANALYSIS: PROBLEM SHEET 7

Problem 1. Compute the following residues.

(a) $\operatorname*{Res}_{z=2\pi\mathbf{i}}\left(\dfrac{1}{z(1-e^{-z})}\right)$, (b) $\operatorname*{Res}_{z=3\pi}(\cot(z))$, (c) $\operatorname*{Res}_{z=0}\left(\dfrac{1}{z^2\sin(z)}\right)$,

(d) $\operatorname*{Res}_{z=n\pi}\left(\dfrac{1}{z^2\sin(z)}\right)$, $(n \in \mathbb{Z}_{\neq 0})$, (e) $\operatorname*{Res}_{z=0}\left(\dfrac{z-\sin(z)}{z}\right)$, (f) $\operatorname*{Res}_{z=0}\left(\dfrac{e^z-e^{-z}}{z^4(1-z^2)}\right)$,

(g) $\operatorname*{Res}_{z=0}\left(\dfrac{\ln(1+z)\sin(z)}{z^5}\right)$.

Problem 2. Prove that $\operatorname*{Res}_{z=\infty}\left(\dfrac{2z^3+7}{z(z-1)^3}\right) = -2$.
(Hint: see problem 4 below).

Problem 3. Let C be the counterclockwise circle of radius 3, centered at 0. Compute the following integral, using the change of variables $w = z^{-1}$: $\displaystyle\int_C \frac{z^3 e^{\frac{1}{z}}}{1+z^3}\,dz$.

Problem 4. Let $n < m$ be two positive integers. Let $P(z) = a_n z^n + a_{n-1}z^{n-1} + \cdots + a_0$ and $Q(z) = b_m z^m + \cdots + b_0$ be two polynomials of degrees n and m respectively. Prove that $\operatorname*{Res}_{z=\infty}\left(\dfrac{P(z)}{Q(z)}\right) = \begin{cases} 0 & \text{if } n < m-1 \\ -\frac{a_n}{b_m} & \text{if } n = m-1 \end{cases}$.
(Hint: we have already solved this problem in Lecture 19.)

Problem 5. Let $\Omega \subset \mathbb{C}$ be an open set and $f : \Omega \to \mathbb{C}$ be a holomorphic function. Assume that $z_0 \in \Omega$ is a zero of f of order $N \in \mathbb{Z}_{\geq 1}$. Prove that $\operatorname*{Res}_{z=z_0}\left(\dfrac{f'(z)}{f(z)}\right) = N$.
(Hint: f vanishes at z_0 to order N means $f(z) = (z-z_0)^N g(z)$, and g does not vanish at all in a small enough disc around z_0 - see Lemma 25.2 and its proof).

Problem 6. Let $\Omega \subset \mathbb{C}$ be an open set, $\alpha \in \Omega$, and $f : \Omega \setminus \{\alpha\} \to \mathbb{C}$ be a holomorphic function, such that α is a pole of f, of order $M \in \mathbb{Z}_{\geq 1}$. Prove that $\operatorname*{Res}_{z=\alpha}\left(\dfrac{f'(z)}{f(z)}\right) = -M$.

Problem 7. Compute the following integrals:

(a) $\displaystyle\int_0^{2\pi} \frac{d\theta}{5+4\sin(\theta)}$, (b) $\displaystyle\int_0^{2\pi} \frac{d\theta}{(a+\cos(\theta))^2}$, $(a \in \mathbb{R}_{>1})$,

(c) $\displaystyle\int_0^{2\pi} \frac{d\theta}{1+\sin^2(\theta)}$, (d) $\displaystyle\int_0^{2\pi} \frac{\cos(2\theta)\,d\theta}{1-2p\cos(\theta)+p^2}$, $(0 < p < 1)$,

Problem 8. For $n \in \mathbb{Z}_{\geq 0}$, prove that $\displaystyle\int_0^{\pi} \sin^{2n}(\theta)\, d\theta = \frac{(2n)!}{2^{2n}(n!)^2}\pi$.

Problem 9. Compute the following integrals.
(a) $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{x^4+1}$, (b) $\displaystyle\int_{-\infty}^{\infty} \frac{x^2}{(x^2+9)(x^2+4)^2}\, dx$.

Problem 10. Let $a, b \in \mathbb{R}_{>0}$ be two positive real numbers. Prove that:

$$\int_0^{\infty} \frac{x^4}{(a+bx^2)^4}\, dx = \frac{\pi}{32a^{\frac{3}{2}}b^{\frac{5}{2}}}.$$

Problem 11. (Bonus) Let $n \in \mathbb{Z}_{\geq 0}$. Prove that

$$\int_0^{2\pi} e^{\cos(\theta)} \cos(n\theta - \sin(\theta))\, d\theta = \frac{2\pi}{n!} \qquad \text{and} \qquad \int_0^{2\pi} e^{\cos(\theta)} \sin(n\theta - \sin(\theta))\, d\theta = 0.$$

(Hint: combine the two integrals as real and imaginary parts of one:

$$\cos(n\theta - \sin(\theta)) + \mathbf{i}\sin(n\theta - \sin(\theta)) = e^{\mathbf{i}(n\theta - \sin(\theta))} = e^{\mathbf{i}n\theta}e^{-\mathbf{i}\sin(\theta)}.$$

Now do the substitution: $z = e^{\mathbf{i}\theta}$.)

Problem 12. (Combine problems 5 and 6 with Cauchy's residue theorem). Let $\Omega \subset \mathbb{C}$ be an open set, and let $f : \Omega \dashrightarrow \mathbb{C}$ be a meromorphic function. (Recall: this means that there is a subset $A \subset \Omega$, such that f is defined and holomorphic on $\Omega \setminus A$; and every point of A is a pole of f).
Let $\gamma : [a, b] \to \Omega$ be a counterclockwise oriented contour, which does not pass through any of the poles of f.

- Let $z_1, z_2, \ldots, z_k \in \text{Interior}(\gamma)$ be zeroes of $f(z)$ which are inside γ, of orders $N_1, N_2, \ldots, N_k$ respectively.

- Let $\alpha_1, \alpha_2, \ldots, \alpha_\ell \in \text{Interior}(\gamma)$ be poles of f which are inside γ, of orders $M_1, M_2, \ldots, M_\ell$ respectively.

Prove that $\displaystyle\frac{1}{2\pi\mathbf{i}}\int_\gamma \frac{f'(z)}{f(z)}\, dz = \sum_{a=1}^{k} N_a - \sum_{b=1}^{\ell} M_b$.