COMPLEX ANALYSIS: PROBLEM SHEET 8

I. Problems from Lecture 28. Problems 1–3 below use Jordan's lemma from §28.1 and Lemma 28.4 to apply the technique of indenting a contour (see Figure 1 below).

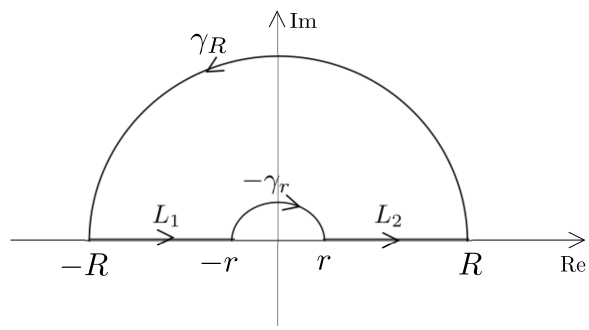

FIGURE 1. Indented contour $C_{r,R}$ consisting of 4 smooth pieces.

Problem 1. Prove that $\displaystyle\int_0^\infty \frac{\sin(x)}{x}\,dx = \frac{\pi}{2}$.

Solution. Since the integrand is even, we have:

$$\int_0^\infty \frac{\sin(x)}{x}\,dx = \frac{1}{2}\,\mathrm{P.V.}\int_{-\infty}^\infty \frac{\sin(x)}{x}\,dx$$

$$= \frac{1}{2}\,\mathrm{Im}\left(\lim_{\substack{R\to\infty\\ r\to 0}}\left(\int_{C_{r,R}} \frac{e^{\mathbf{i}z}}{z}\,dz - \int_{\gamma_R}\frac{e^{\mathbf{i}z}}{z}\,dz + \int_{\gamma_r}\frac{e^{\mathbf{i}z}}{z}\,dz\right)\right)$$

Step 1. Since $\dfrac{e^{\mathbf{i}z}}{z}$ does not have any singularities within $C_{r,R}$, by Cauchy's theorem $\displaystyle\int_{C_{r,R}} \frac{e^{\mathbf{i}z}}{z}\,dz = 0$.

Step 2. On γ_R, $\left|\dfrac{1}{z}\right| = \dfrac{1}{R}$ which goes to 0 as $R \to \infty$. Hence, by Jordan's lemma: $\displaystyle\lim_{R\to\infty}\int_{\gamma_R}\frac{e^{\mathbf{i}z}}{z}\,dz = 0$.

Step 3. As the pole of $\dfrac{e^{\mathbf{i}z}}{z}$ at $z = 0$ is of order 1, Lemma 28.4 applies, and we get:

$$\lim_{r\to 0}\int_{\gamma_r}\frac{e^{\mathbf{i}z}}{z}\,dz = \pi\mathbf{i}\operatorname*{Res}_{z=0}\left(\frac{e^{\mathbf{i}z}}{z}\right) = \pi\mathbf{i}.$$

Hence,

$$\int_0^\infty \frac{\sin(x)}{x}\,dx = \frac{1}{2}\,\mathrm{Im}(\pi\mathbf{i}) = \frac{\pi}{2}.$$

Problem 2. Prove that $\displaystyle\int_0^\infty \frac{\sin^3(x)}{x^3}\,dx = \frac{3\pi}{8}$.

Solution. Let $f(z) = e^{3\mathbf{i}z} - 3e^{\mathbf{i}z} + 2$. Note that for $x \in \mathbb{R}$:

$$\operatorname{Im}(f(x)) = \sin(3x) - 3\sin(x) = \sin(2x)\cos(x) + \cos(2x)\sin(x) - 3\sin(x)$$
$$= 2\sin(x)\cos^2(x) + (1-2\sin^2(x))\sin(x) - 3\sin(x) = 2\sin(x)\cos^2(x) - 2\sin^3(x) - 2\sin(x)$$
$$= -2\sin(x)(1-\cos^2(x)) - 2\sin^3(x) = -4\sin^3(x).$$

Again, the function $\dfrac{\sin^3(x)}{x^3}$ is even, so we get:

$$\int_0^\infty \frac{\sin^3(x)}{x^3}\,dx = \frac{1}{2}\,\text{P. V.}\int_{-\infty}^\infty \frac{\sin^3(x)}{x^3}\,dx$$
$$= -\frac{1}{8}\operatorname{Im}\left(\lim_{\substack{R\to\infty\\ r\to 0}}\left(\int_{C_{r,R}} \frac{f(z)}{z^3}\,dz - \int_{\gamma_R} \frac{f(z)}{z^3}\,dz + \int_{\gamma_r} \frac{f(z)}{z^3}\,dz\right)\right)$$

Step 1. Since $\dfrac{f(z)}{z^3}$ has no singularities within $C_{r,R}$, by Cauchy's theorem $\displaystyle\int_{C_{r,R}} \frac{f(z)}{z^3}\,dz = 0$.

Step 2. On γ_R, $\left|\dfrac{1}{z^3}\right| = \dfrac{1}{R^3}$. By Jordan's lemma $\displaystyle\lim_{R\to\infty} \frac{e^{3\mathbf{i}z}}{z^3}\,dz = 0$ and $\displaystyle\lim_{R\to\infty} \frac{-3e^{\mathbf{i}z}}{z^3}\,dz = 0$. For the last term of $f(z)$, we use the important inequality:

$$\left|\int_{\gamma_R} \frac{2}{z^3}\,dz\right| \leq \frac{2}{R^3}\cdot \pi R = \frac{2\pi}{R^2} \to 0 \text{ as } R \to \infty.$$

Hence, $\displaystyle\lim_{R\to\infty} \frac{f(z)}{z^3}\,dz = 0$.

Step 3. The function $\dfrac{f(z)}{z^3}$ can be expanded using the Taylor series expansion of e^z:

$$f(z) = e^{3\mathbf{i}z} - 3e^{\mathbf{i}z} + 2 = (1-3+2) + z\,(3\mathbf{i} - 3\mathbf{i}) + z^2\left(\frac{(3\mathbf{i})^2}{2} - 3\frac{\mathbf{i}^2}{2}\right) + \cdots$$
$$= -3z^2 + \cdots$$

Hence $\dfrac{f(z)}{z^3}$ has a pole of order 1 at $z = 0$, and $\operatorname{Res}_{z=0}\left(\frac{f(z)}{z^3}\right) = -3$. Using Lemma 28.4, we have:

$$\lim_{r\to 0}\int_{\gamma_r} \frac{f(z)}{z^3}\,dz = \pi\mathbf{i}\operatorname{Res}_{z=0}\left(\frac{f(z)}{z^3}\right) = -3\pi\mathbf{i}.$$

Combining all these, we get:

$$\int_0^\infty \frac{\sin^3(x)}{x^3}\,dx = -\frac{1}{8}\operatorname{Im}(-3\pi\mathbf{i}) = \frac{3\pi}{8}.$$

Problem 3. Let $a, b \in \mathbb{R}_{>0}$. Prove that $\displaystyle\int_0^\infty \frac{\cos(2ax) - \cos(2bx)}{x^2}\,dx = \pi(b-a)$.

Solution. The integrand is again even. So,

$$\int_0^\infty \frac{\cos(2ax) - \cos(2bx)}{x^2}\,dx = \frac{1}{2}\,\mathrm{P.V.}\int_{-\infty}^\infty \frac{\cos(2ax) - \cos(2bx)}{x^2}\,dx$$

$$= \frac{1}{2}\,\mathrm{Re}\left(\lim_{\substack{R\to\infty \\ r\to 0}}\left(\int_{C_{r,R}} \frac{e^{2a\mathbf{i}z} - e^{2b\mathbf{i}z}}{z^2}\,dz - \int_{\gamma_R} \frac{e^{2a\mathbf{i}z} - e^{2b\mathbf{i}z}}{z^2}\,dz + \int_{\gamma_r} \frac{e^{2a\mathbf{i}z} - e^{2b\mathbf{i}z}}{z^2}\,dz\right)\right)$$

Step 1. Again the integrand is holomorphic within $C_{r,R}$, so by Cauchy's theorem $\displaystyle\int_{C_{r,R}} \frac{e^{2a\mathbf{i}z} - e^{2b\mathbf{i}z}}{z^2}\,dz = 0$.

Step 2. On γ_R, $\left|\dfrac{1}{z^2}\right| = \dfrac{1}{R^2} \to 0$, as $R \to \infty$. So Jordan's lemma applies, and we get $\displaystyle\lim_{R\to\infty}\int_{\gamma_R} \frac{e^{2a\mathbf{i}z} - e^{2b\mathbf{i}z}}{z^2}\,dz = 0$.

Step 3. Use the Taylor series expansion:

$$e^{2a\mathbf{i}z} - e^{2b\mathbf{i}z} = 2\mathbf{i}(a-b)z + \cdots$$

which implies that $\dfrac{e^{2a\mathbf{i}z} - e^{2b\mathbf{i}z}}{z^2}$ has a pole of order 1 at $z = 0$, with residue $= 2\mathbf{i}(a-b)$. By Lemma 28.4:

$$\lim_{r\to 0}\int_{\gamma_r} \frac{e^{2a\mathbf{i}z} - e^{2b\mathbf{i}z}}{z^2}\,dz = \pi\mathbf{i}\operatorname*{Res}_{z=0}\left(\frac{e^{2a\mathbf{i}z} - e^{2b\mathbf{i}z}}{z^2}\right) = -2\pi(a-b) = 2\pi(b-a).$$

Combining all this, we get:

$$\int_0^\infty \frac{\cos(2ax) - \cos(2bx)}{x^2}\,dx = \frac{1}{2}\,\mathrm{Re}(2\pi(b-a)) = \pi(b-a).$$

II. Problems from Lecture 30. Problems 4–6 below involve the Laplace transform. See Lecture 30, §30.5.

Problem 4. Let $n \in \mathbb{Z}_{\geq 0}$ and $\varphi(t) = \dfrac{t^n}{n!}$. Prove that the Laplace transform $\mathcal{L}\varphi(z) = z^{-n-1}$, if $\mathrm{Re}(z) > 0$.

Solution. Let $f_n(z) = \displaystyle\int_0^\infty \frac{t^n}{n!}e^{-zt}\,dt$.

$$f_0(z) = \left[\frac{e^{-zt}}{-z}\right]_{t=0}^\infty = \frac{1}{z} - \frac{1}{z}\lim_{t\to\infty} e^{-zt}$$

The last limit is zero since $\mathrm{Re}(z) > 0$ (recall $|e^{-zt}| = e^{-\mathrm{Re}(z)t}$). Hence $f_0(z) = z^{-1}$.

Now, we have, for every $n \geq 0$:

$$f_{n+1}(z) = \int_0^\infty \frac{t^{n+1}}{(n+1)!} e^{-zt}\, dt = \left[\frac{t^{n+1}}{(n+1)!} \frac{e^{-zt}}{-z} \right]_{t=0}^{\infty} + \frac{1}{z} \int_0^\infty \frac{t^n}{n!} e^{-zt}\, dt$$

The first term is 0 because $t^{n+1}e^{-zt}|_{t=0} = 0$ and (since $\mathrm{Re}(z) > 0$: $\lim_{t\to\infty} t^{n+1}e^{-zt} = 0$. Hence, we get:

$$f_{n+1}(z) = z^{-1} f_n(z) \Rightarrow f_{n+1}(z) = z^{-n-1} f_0(z) = z^{-n-2}.$$

Problem 5. Let $\varphi(t) = e^t$. Prove that $\mathcal{L}\varphi(z) = \dfrac{1}{z-1}$, if $\mathrm{Re}(z) > 1$.

Solution. $\mathcal{L}\varphi(z) = \int_0^\infty e^t e^{-zt}\, dt = \left[\dfrac{e^{-(z-1)t}}{-(z-1)} \right]_{t=0}^{\infty}$. For $\mathrm{Re}(z) > 1$, we have $\lim_{t\to\infty} e^{-(z-1)t} = 0$. Hence, we get $\mathcal{L}\varphi(z) = \dfrac{1}{z-1}$.

Problem 6. Let $\varphi(t)$ be a continuous function of a real variable $t \in \mathbb{R}_{\geq 0}$. Let $c \in \mathbb{R}$ and define $\psi(t) = e^{ct}\varphi(t)$. Prove that $\mathcal{L}\psi(z) = \mathcal{L}\varphi(z-c)$.
Solution.

$$\mathcal{L}\psi(z) = \int_0^\infty \psi(t) e^{-zt}\, dt = \int_0^\infty e^{ct}\varphi(t) e^{-zt}\, dt$$
$$= \int_0^\infty \varphi(t) e^{-(z-c)t}\, dt = \mathcal{L}\varphi(z-c).$$

III. Problems from Lectures 31, 32 - Gamma function.

Problem 7. Prove that, for every $n \in \mathbb{Z}_{\geq 0}$, we have $\operatorname*{Res}_{z=-n}(\Gamma(z)) = \dfrac{(-1)^n}{n!}$.
Solution. Given in Lecture 31, §31.4, Example on page 5.

Problem 8. Consider the function (sometimes called Gauss' Psi-function)

$$\boxed{\Psi(z) = \frac{\Gamma'(z)}{\Gamma(z)}}$$

(a) Verify the following formula, using Weierstrass' formula for the Gamma function:

$$\Psi(z) = -\gamma - \frac{1}{z} - \sum_{n=1}^{\infty} \left(\frac{1}{z+n} - \frac{1}{n} \right) .$$

Solution. $\Gamma(z)$ is defined by the following formula:

$$\frac{1}{\Gamma(z)} = z e^{\gamma z} \prod_{k=1}^{\infty} \left\{ \left(1 + \frac{z}{k}\right) e^{-\frac{z}{k}} \right\}$$

Thus, $\dfrac{1}{\Gamma(z)}$ is the uniform limit of the following sequence of functions $\{F_N(z)\}_{N=1}^{\infty}$:

$$F_N(z) = ze^{\gamma z}\prod_{k=1}^{N}\left\{\left(1+\frac{z}{k}\right)e^{-\frac{z}{k}}\right\}$$

So, $-\dfrac{\Gamma'(z)}{\Gamma(z)}$ is the uniform limit of $\dfrac{F_N'(z)}{F_N(z)}$, which we can compute as:

$$\frac{F_N'(z)}{F_N(z)} = \frac{1}{z} + \gamma + \sum_{k=1}^{N}\left(\frac{1}{z+k} - \frac{1}{k}\right)$$

This allows us to write $\Psi(z) = \dfrac{\Gamma'(z)}{\Gamma(z)}$ as:

$$\boxed{\Psi(z) = -\frac{1}{z} - \gamma - \sum_{k=1}^{\infty}\left(\frac{1}{z+k} - \frac{1}{k}\right)}$$

(b) Use (a) to prove that $\Psi(z+1) - \Psi(z) = \dfrac{1}{z}$. Deduce that $\Gamma(z+1) = z\Gamma(z)$.
Solution. $\Psi(z)$ is the uniform limit of the following sequence of functions:

$$S_N(z) = -\frac{1}{z} - \gamma - \sum_{k=1}^{N}\left(\frac{1}{z+k} - \frac{1}{k}\right)$$

Let us compute $S_N(z+1) - S_N(z)$:

$$S_N(z+1) - S_N(z) = -\frac{1}{z+1} + \frac{1}{z} - \sum_{k=1}^{N}\left(\frac{1}{z+1+k} - \frac{1}{z+k}\right)$$

$$= -\frac{1}{z+1} + \frac{1}{z} - \left(\frac{1}{z+2} - \frac{1}{z+1}\right) - \left(\frac{1}{z+3} - \frac{1}{z+2}\right) - \cdots - \left(\frac{1}{z+N+1} - \frac{1}{z+N}\right)$$

$$= \frac{1}{z} - \frac{1}{z+N+1}$$

As $N \to \infty$, the last term approaches $\dfrac{1}{z}$. Hence, we get:

$$\boxed{\Psi(z+1) - \Psi(z) = \frac{1}{z}}$$

(c) Prove that $\Psi(z) - \Psi(1-z) = -\pi\cot(\pi z)$.
Solution. Recall that we have $\Gamma(z)\Gamma(1-z) = \dfrac{\pi}{\sin(\pi z)}$. Taking logarithmic derivative gives:

$$\frac{\Gamma'(z)}{\Gamma(z)} - \frac{\Gamma'(1-z)}{\Gamma(1-z)} = -\frac{d}{dz}(\log(\sin(\pi z))) = -\pi\frac{\cos(\pi z)}{\sin(\pi z)}$$

That is,

$$\Psi(z) - \Psi(1-z) = -\pi\cot(\pi z).$$

Problem 9. Let $n \in \mathbb{Z}_{\geq 1}$. Prove that $\Gamma'(n) = (n-1)!\left(-\gamma + \sum_{k=1}^{n-1} \frac{1}{k}\right)$.

Solution. We can compute $\Psi(1)$ as follows.

$$S_N(1) = -1 - \gamma - \sum_{k=1}^{N} \left(\frac{1}{1+k} - \frac{1}{k}\right)$$

$$= -1 - \gamma - \left(\frac{1}{2} - 1\right) - \left(\frac{1}{3} - \frac{1}{2}\right) - \cdots - \left(\frac{1}{N+1} - \frac{1}{N}\right)$$

$$= -\gamma - \frac{1}{N+1} \to -\gamma \text{ as } N \to \infty$$

Hence, $\Psi(1) = -\gamma$. In the equation $\Psi(z+1) = \Psi(z) + \frac{1}{z}$, set $z = n-1$, to get:

$$\Psi(n) = \Psi(n-1) + \frac{1}{n-1}.$$

This gives $\Psi(n) = -\gamma + \sum_{k=0}^{n-1} \frac{1}{k}$. Since $\Gamma'(n) = \Gamma(n)\Psi(n) = (n-1)!\Psi(n)$, the equation given in the problem above follows.

Problem 10. Use $\Psi(z)$ from Problem 8 to find a function $F(z)$ such that:

$$F(z+1) = F(z) - \frac{1}{(z-2)^2} \ .$$

Solution. Since $\Psi(z+1) = \Psi(z) + \frac{1}{z}$, taking its derivative, we get:

$$\Psi'(z+1) = \Psi'(z) - \frac{1}{z^2}.$$

So, the given equation is solved by $F(z) = \Psi'(z-2)$.

More explicitly (or directly), $\Psi'(z) = \sum_{k=0}^{\infty} \frac{1}{(z+k)^2}$. So $F(z) = \sum_{k=0}^{\infty} \frac{1}{(z-2+k)^2} = \sum_{\ell=-2}^{\infty} \frac{1}{(z+\ell)^2}$.

Problem 10***.** Let $P(z), Q(z)$ be two polynomials. Obtain a method to solve the equation: $F(z+1) - F(z) = \frac{P(z)}{Q(z)}$.

Solution. $\Psi(z)$ and its derivatives can help us solve any difference equation of the type:

$$F(z+1) = F(z) + R(z)$$

where $R(z)$ is a rational function. Assume first that $R(z) = \frac{P(z)}{Q(z)}$, with degree of $P <$ degree of Q. Writing in terms of its partial fractions, it suffices to consider the following equation:

$$F(z+1) = F(z) + \frac{1}{(z-a)^{\ell+1}} ,$$

where $a \in \mathbb{C}$ and $\ell \in \mathbb{Z}_{\geq 0}$.

For $\ell = 0$, this equation is solved by $\Psi(z-a)$. In general, taking derivatives of the difference equation satisfied by $\Psi(z)$, we have:

$$\Psi^{(\ell)}(z+1) = \Psi^{(\ell)}(z) + (-1)^{\ell}\frac{\ell!}{z^{\ell+1}}.$$

Hence, $F(z) = (-1)^{\ell}\frac{\Psi^{(\ell)}(z-a)}{\ell!}$ solves the equation $F(z+1) = F(z) + \frac{1}{(z-a)^{\ell+1}}$.

If $R(z) = \frac{P(z)}{Q(z)}$ with degree of $P \geq$ degree of Q, then we can use polynomial division to get $R(z) = p(z) + \frac{P_1(z)}{Q(z)}$ where $p(z)$ is a polynomial and degree of $P_1 <$ degree of Q.

This reduces problem 10* to the following: given a polynomial $p(z)$, say of degree n, find $f(z)$ such that

$$f(z+1) - f(z) = p(z) .$$

We don't need any special functions to solve this one. You can solve it with $f(z)$ also a polynomial (of degree $n+1$). Here is a trick to do this. (it is called "discrete integration"). First of all, define:

$$m_0(z) = 1, \qquad m_n(z) = z(z-1)(z-2)\cdots(z-n+1) \text{ for } n \geq 1.$$

It is very easy (by induction) to prove that any polynomial $p(z)$ of degree n can be written in terms of $\{m_0(z), m_1(z), \ldots, m_n(z)\}$. This reduces the problem to finding $f(z)$ such that $f(z+1) - f(z) = m_n(z)$ $(n \geq 0)$. Now it is almost trivial to check that $f(z) = \frac{1}{n+1}m_{n+1}(z)$ solves this equation.

(To justify the name "discrete integration" compare with the fact that $g(x) = \frac{x^{n+1}}{n+1}$ solves the equation $g'(x) = x^n$. If we were to replace derivative by difference operator $f(z) \mapsto f(z+1) - f(z)$, then the role of integration will be played by $m_n(z) \mapsto \frac{m_{n+1}(z)}{n+1}$.)

Problem 11. Use $\Gamma(z)$ to solve the following equation:

$$F(z+1) = \frac{z^2 - 2z}{(z+\mathbf{i})^3}F(z) .$$

Solution. $F(z) = \frac{\Gamma(z)\Gamma(z-2)}{\Gamma(z+\mathbf{i})^3}$ solves this equation.

Problem 11*. Let $P(z), Q(z)$ be two polynomials. Obtain a solution of

$$F(z+1) = \frac{P(z)}{Q(z)} F(z).$$

Solution. Writing in terms of their roots, we can assume that the rational function is of the form:

$$\frac{P(z)}{Q(z)} = C \frac{\prod_{i=1}^{n}(z-a_i)}{\prod_{j=1}^{m}(z-b_j)}$$

where $C \in \mathbb{C}$ is a non–zero complex number. Choose a complex number $c \in \mathbb{C}$ such that $e^c = C$. Then, our solution is:

$$F(z) = e^{cz} \frac{\prod_{i=1}^{n}\Gamma(z-a_i)}{\prod_{j=1}^{m}\Gamma(z-b_j)}$$

Bonus 1. Let $y \in \mathbb{R}_{\neq 0}$. Prove that

$$|\Gamma(\mathbf{i}y)| = \sqrt{\frac{2\pi}{y(e^{\pi y} - e^{-\pi y})}}$$

Solution. Using the definition of the gamma function, we have

$$\frac{1}{\Gamma(\mathbf{i}y)} = (\mathbf{i}y)e^{\mathbf{i}\gamma y} \prod_{n=1}^{\infty} \left\{ \left(1 + \frac{\mathbf{i}y}{n}\right) e^{-\frac{\mathbf{i}y}{n}} \right\}$$

(here $y \in \mathbb{R}_{\neq 0}$). Taking modulus square on both sides gives us:

$$\frac{1}{|\Gamma(\mathbf{i}y)|^2} = y^2 \prod_{n=1}^{\infty} \left(1 + \frac{y^2}{n^2}\right)$$

Now, use the fact that $\dfrac{\sin(z)}{z} = \displaystyle\prod_{n=1}^{\infty}\left(1 - \frac{z^2}{n^2\pi^2}\right)$ from Lecture 29, §29.0, to write:

$$\prod_{n=1}^{\infty}\left(1 + \frac{y^2}{n^2}\right) = \frac{\sin(\pi \mathbf{i} y)}{\pi \mathbf{i} y} = \frac{e^{\pi y} - e^{-\pi y}}{2\pi y}$$

which implies:

$$|\Gamma(\mathbf{i}y)|^2 = \frac{2\pi}{y(e^{\pi y} - e^{-\pi y})}.$$

Bonus 2.[1] Let $f(t)$ be a continuous function of one real variable t. Let T be the triangular region in $\mathbb{R}^2$, given by:

$$T = \{(x, y) : x \geq 0, y \geq 0 \text{ and } x + y \leq 1\}$$

Prove that, for any two $a, b \in \mathbb{R}_{>0}$, we have:

$$\iint_T f(x+y)x^{a-1}y^{b-1}\,dxdy = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} \int_0^1 f(t)t^{a+b-1}\,dt$$

[1]This fascinating computation is due to Dirichlet, around 1829.

Solution. (1) Let us write this integral as:

$$\int_0^1 \left(\int_0^{1-x} f(x+y) x^{a-1} y^{b-1} dy \right) dx \ .$$

(2) Perform the change of variables: $y = x(1-v)/v$. This turns our integral into:

$$\int_0^1 \left(\int_x^1 f\left(\frac{x}{v}\right) \frac{x^{a+b-1}}{v^{b+1}} (1-v)^{b-1} \, dv \right) dx$$

(3) Flipping the order of the integration, we can write this as:

$$\int_0^1 \left(\int_0^v f\left(\frac{x}{v}\right) \frac{x^{a+b-1}}{v^{b+1}} (1-v)^{b-1} \, dx \right) dv$$

(4) Change of variables: $vt = x$ turns this into:

$$\int_0^1 \left(\int_0^1 f(t) t^{a+b-1} v^{a-1} (1-v)^{b-1} \, dx \right) dv$$

Using Euler's formula given in Lecture 31, §31.6, $\displaystyle\int_0^1 v^{a-1}(1-v)^{b-1}\, dv = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$. Hence, we get:

$$\iint_T f(x+y) x^{a-1} y^{b-1} \; dxdy = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} \int_0^1 f(t) t^{a+b-1} \, dt$$