COMPLEX ANALYSIS: PROBLEM SHEET 8

I. Problems from Lecture 28. Problems 1–3 below use Jordan's lemma from §28.1 and Lemma 28.4 to apply the technique of indenting a contour (see Figure 1 below).

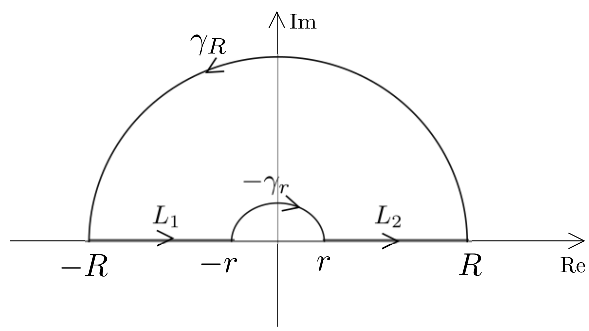

FIGURE 1. Indented contour $C_{r,R}$ consisting of 4 smooth pieces.

Problem 1. Prove that $\displaystyle\int_0^\infty \frac{\sin(x)}{x}\,dx = \frac{\pi}{2}$.

Problem 2. Prove that $\displaystyle\int_0^\infty \frac{\sin^3(x)}{x^3}\,dx = \frac{3\pi}{8}$.
(Hint: consider $f(z) = e^{3\mathbf{i}z} - 3e^{\mathbf{i}z} + 2$. Verify that for $x \in \mathbb{R}$, $\mathrm{Im}(f(x)) = -4\sin^3(x)$ and that $\dfrac{f(z)}{z^3}$ has a pole of order 1 at $z = 0$, so Lemma 28.4 applies.)

Problem 3. Let $a, b \in \mathbb{R}_{>0}$. Prove that $\displaystyle\int_0^\infty \frac{\cos(2ax) - \cos(2bx)}{x^2}\,dx = \pi(b-a)$.

II. Problems from Lecture 30. Problems 4–6 below involve the Laplace transform. See Lecture 30, §30.5.

Problem 4. Let $n \in \mathbb{Z}_{\geq 0}$ and $\varphi(t) = \dfrac{t^n}{n!}$. Prove that the Laplace transform $\mathcal{L}\varphi(z) = z^{-n-1}$, if $\mathrm{Re}(z) > 0$.

Problem 5. Let $\varphi(t) = e^t$. Prove that $\mathcal{L}\varphi(z) = \dfrac{1}{z-1}$, if $\mathrm{Re}(z) > 1$.

Problem 6. Let $\varphi(t)$ be a continuous function of a real variable $t \in \mathbb{R}_{\geq 0}$. Let $c \in \mathbb{R}$ and define $\psi(t) = e^{ct}\varphi(t)$. Prove that $\mathcal{L}\psi(z) = \mathcal{L}\varphi(z-c)$.

III. Problems from Lectures 31, 32 - Gamma function. For the rest of the problems, you can use the following facts.

(1) Definition of $\Gamma(z)$ (Weierstrass). $\Gamma : \mathbb{C} \dashrightarrow \mathbb{C}$ is a meromorphic function with poles of order 1 at $z = 0, -1, -2, \ldots$

$$\Gamma(z) = \frac{1}{ze^{\gamma z}} \prod_{n=1}^{\infty} \left\{ \left(1 + \frac{z}{n}\right)^{-1} e^{\frac{z}{n}} \right\}$$

where γ is the Euler–Mascheroni constant (see Lecture 32, §32.1–32.3).

(2) For $\mathrm{Re}(z) > 0$, $\Gamma(z) = \displaystyle\int_0^{\infty} t^{z-1} e^{-t} \, dt$ (Euler's integral).

(3) $\Gamma(z+1) = z\Gamma(z)$. $\Gamma(n) = (n-1)!$, for $n \in \mathbb{Z}_{\geq 1}$. (These facts were proved in Lecture 31, §31.3, 31.4, using Euler's integral. It will be beneficial to prove these directly from Weierstrass' formula - see Problem 8 below.)

(4) $\Gamma(z)\Gamma(1-z) = \dfrac{\pi}{\sin(\pi z)}$ (see Lecture 32, §32.5).

Problem 7. Prove that, for every $n \in \mathbb{Z}_{\geq 0}$, we have $\operatorname*{Res}_{z=-n} (\Gamma(z)) = \dfrac{(-1)^n}{n!}$.

Problem 8. Consider the function (sometimes called Gauss' Psi-function)

$$\boxed{\Psi(z) = \frac{\Gamma'(z)}{\Gamma(z)}}$$

(a) Verify the following formula, using Weierstrass' formula for the Gamma function:

$$\Psi(z) = -\gamma - \frac{1}{z} - \sum_{n=1}^{\infty} \left(\frac{1}{z+n} - \frac{1}{n} \right) .$$

(b) Use (a) to prove that $\Psi(z+1) - \Psi(z) = \dfrac{1}{z}$. Deduce that $\Gamma(z+1) = z\Gamma(z)$.
This proves that Weierstrass' formula also solves $F(z+1) = zF(z)$. Unlike the proof in the notes, this proof avoids the use of Euler's integral.

(c) Prove that $\Psi(z) - \Psi(1-z) = -\pi \cot(\pi z)$.

Problem 9. Let $n \in \mathbb{Z}_{\geq 1}$. Prove that $\Gamma'(n) = (n-1)! \left(-\gamma + \displaystyle\sum_{k=1}^{n-1} \frac{1}{k} \right)$.

(Hint: use Problem 8 (a).)

Problem 10. Use $\Psi(z)$ from Problem 8 to find a function $F(z)$ such that:

$$F(z+1) = F(z) - \frac{1}{(z-2)^2} .$$

Problem 10*. Let $P(z), Q(z)$ be two polynomials. Obtain a method to solve the equation: $F(z+1) - F(z) = \dfrac{P(z)}{Q(z)}$.

Problem 11. Use $\Gamma(z)$ to solve the following equation:

$$F(z+1) = \frac{z^2 - 2z}{(z+\mathbf{i})^3} F(z) \ .$$

Problem 11*. Let $P(z), Q(z)$ be two polynomials. Obtain a solution of

$$F(z+1) = \frac{P(z)}{Q(z)} F(z).$$

Bonus 1. Let $y \in \mathbb{R}_{\neq 0}$. Prove that

$$|\Gamma(\mathbf{i}y)| = \sqrt{\frac{2\pi}{y(e^{\pi y} - e^{-\pi y})}}$$

(Hint: use the expansion of $\dfrac{\sin(z)}{z}$ from Lecture 29, page 1, and the fact that $\sin(\mathbf{i}y) = \mathbf{i}(e^y - e^{-y})/2$.)

Bonus 2.[1] Let $f(t)$ be a continuous function of one real variable t. Let T be the triangular region in $\mathbb{R}^2$, given by:

$$T = \{(x, y) : x \leq 0, y \leq 0 \text{ and } x + y \leq 1\}$$

Prove that, for any two $a, b \in \mathbb{R}_{>0}$, we have:

$$\iint_T f(x+y) x^{a-1} y^{b-1} \, dx dy = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} \int_0^1 f(t) t^{a+b-1} \, dt$$

Hint: (1) Write this integral as:

$$\int_0^1 \left(\int_0^{1-x} f(x+y) x^{a-1} y^{b-1} dy \right) dx \ .$$

(2) Perform the change of variables: $y = x(1-v)/v$.
(3) Flip the order of integration.
(4) Change variables again: $vt = x$.
(5) Use the result from Lecture 31, §31.6.

[1] This fascinating computation is due to Dirichlet, around 1829.