COMPLEX ANALYSIS: OPTIONAL READING A

(A.0) What is in these notes.– These notes contain proofs of some of the well–known facts from real analysis, that we have used in our course.

(1) Cauchy's criterion for existence of the limit of a sequence of numbers (mentioned in Lecture 21, page 3) is proved below in §A.4. This proof uses Bolzano–Weierstrass theorem §A.2.

(2) A descending chain of closed, bounded intervals, whose lengths approach 0, intersect at a unique point §A.5. This fact was used in our proof of Cauchy's theorem for rectangles: Lecture 15, page 4.

(3) A continuous function on a closed, bounded set always attains its maximum and minimum values: §A.6.

(4) We claimed in Lecture 13, page 8, that if $\Omega \subset \mathbb{C}$ is an open and connected set, then any two points can be connected by a zig-zag path. The proof uses Heine–Borel theorem given in §A.7 below.

    Proof. Let $\alpha, \beta \in \Omega$. As Ω is connected we can find a continuous $\gamma : [a, b] \to \Omega$ such that $\gamma(a) = \alpha$ and $\gamma(b) = \beta$. For every point $\gamma(t) \in \Omega$ we can find an open disc around $\gamma(t)$ of radius $r(t) > 0$ which is still inside Ω: $D(\gamma(t); r(t)) \subset \Omega$ (since Ω is open). Take $\varepsilon(t) > 0$ be such that γ maps the open interval $(t - \varepsilon(t), t + \varepsilon(t))$ inside this open disc $D(\gamma(t), r(t))$ (this can be done, since γ is continuous).
    Now we have covered $[a, b]$ by open intervals $\{I_t = (t - \varepsilon(t), t + \varepsilon(t))\}_{t\in[a,b]}$. By Theorem A.7, it is possible to choose a finite subcollection which also covers $[a, b]$. That is, finitely many of these discs suffice. Within each disc, it is possible to replace γ by a zig-zag (see the picture at the end of page 8 of Lecture 13).
    $\square$

(A.1) Completeness axiom of real line.– Recall that $\mathbb{R}$ denotes the set of real numbers. The following property of $\mathbb{R}$ is to be taken as an axiom (called the completeness axiom of real line):

Given any subset $A \subset \mathbb{R}$ which is bounded above (that is, there exists some number $c \in \mathbb{R}$ such that $a < c$ for every $a \in A$), the *supremum* (also called the *least upper bound*) of A exists. Often denoted by $\sup(A)$, this is the unique real number such that:

- $a \leq \sup(A)$ for every $a \in A$.
- If $c \in \mathbb{R}$ is such that $a \leq c$ for every $a \in A$, then $\sup(A) \leq c$.

Note that $\sup(A)$ has to be unique (if there are two such numbers, say s_1 and s_2, then $s_1 \leq s_2$ by the second property applied to s_1; and $s_2 \leq s_1$ by the same property applied to s_2).

Similarly, for a subset $B \subset \mathbb{R}$ which is bounded below we have the existence and uniqueness of $\inf(B)$. Namely: $\inf(B) = -\sup(-B)$.

Notation for intervals. For $a < b$, we have: (i) the open interval $(a, b) = \{x \in \mathbb{R} : a < x < b\}$; (ii) the closed interval $[a, b] = \{x \in \mathbb{R} : a \leq x \leq b\}$; (iii) $[a, b) = \{x \in \mathbb{R} : a \leq x < b\}$, and so on.

(A.2) Bolzano–Weierstrass theorem.– [1] Consider a sequence $\{x_n\}_{n=0}^{\infty}$ of real numbers.

Definition. A number $c \in \mathbb{R}$ is called a *cluster point* of $\{x_n\}_{n=0}^{\infty}$ if for every $\varepsilon > 0$, there are infinitely many $x_n's$ in $(c - \varepsilon, c + \varepsilon)$.

Theorem. *Let $\{x_n\}_{n=0}^{\infty}$ be a sequence of real numbers which is bounded (that is, there exist real numbers $L < R$ so that $x_n \in [L, R]$ for every $n \geq 0$). Then the sequence has at least one cluster point.*

Proof. For each $k \in \mathbb{Z}_{\geq 0}$ consider the subsequence $\{x_n\}_{n=k}^{\infty}$. It has a least upper bound by the completeness axiom. Let $\ell_k = \sup(\{x_k, x_{k+1}, \ldots\})$. This is clearly a decreasing sequence of numbers, all bigger than L:

$$\ell_0 \geq \ell_1 \geq \ell_2 \geq \ell_3 \geq \cdots \geq L.$$

Take $G = \inf(\{\ell_0, \ell_1, \ell_2, \ldots\})$. We are going to prove that G is a cluster point of $\{x_n\}_{n=0}^{\infty}$. (see next paragraph: this G is the limit-supremum of $\{x_n\}_{n=0}^{\infty}$.)

So, let $\varepsilon > 0$ be given. As G is the infimum of $\{\ell_k\}_{k=0}^{\infty}$, and $G + \varepsilon > G$, there must be some N so that $\ell_N < G + \varepsilon$ (if all $\ell_k's$ are larger than $G + \varepsilon$, then $G + \varepsilon$ has to be smaller than, or equal to the infimum). Since $\ell_N = \sup(\{x_N, x_{N+1}, \ldots\})$, we conclude that $x_n < G + \varepsilon$ for every $n \geq N$.

Now we show that infinitely many of $x_n's$ are larger than $G - \varepsilon$. We will prove it by contradiction. So, assume that only finitely many $x_n's$ are larger than $G - \varepsilon$. That means, there is some positive integer M, so that $x_n < G - \varepsilon$ for every $n \geq M$. But that means:

$$\ell_M = \sup(\{x_M, x_{M+1}, \ldots\}) \leq G - \varepsilon < G \leq \ell_M.$$

This is a contradiction, and we are done.

□

(A.3) $\overline{\lim}$ and $\underline{\lim}$.– The definition of *limit-supremum* was given in Lecture 21, §21.2, page 3. We recall it below.

Definition. Given a sequence $\{x_n\}_{n=0}^{\infty}$ of real number, we say $\boxed{G = \overline{\lim_{n\to\infty}} x_n}$ (in words: G is the limit-supremum, or lim-sup, of the sequence $\{x_n\}$) if it has the following property: for every $\varepsilon > 0$, we have:

[1] This theorem, often attributed to Weierstrass, was proved by Bolzano in 1817. Bernard Bolzano (1781-1848) wrote it as a lemma required in his proof of the intermediate value theorem.

- There are infinitely many $x_n's$ in the interval $(G - \varepsilon, G + \varepsilon)$.
- Almost all of $x_n's$ are less than $G + \varepsilon$. Meaning, there is some number $N > 0$, such that $x_n < G + \varepsilon$ for every $n \geq N$.

In the proof of Bolzano–Weierstrass theorem given above, we constructed the limit-supremum of a bounded sequence of real numbers:

$$\boxed{\overline{\lim_{n\to\infty}} x_n = \inf_k(\sup(\{x_k, x_{k+1}, \ldots\}))}$$

Similarly, limit-infimum, or lim-inf, can be defined as follows. $L = \underline{\lim_{n\to\infty}} x_n$ means that for every $\varepsilon > 0$, we have:

- There are infinitely many $x_n's$ in the interval $(L - \varepsilon, L + \varepsilon)$.
- Almost all of $x_n's$ are greater than $L - \varepsilon$. That is to say, there is some number $N > 0$, such that $x_n > L - \varepsilon$ for every $n \geq N$.

Lim-inf is constructed for a bounded sequence by flipping the order of inf and sup in the construction of lim-sup above. Or: $\underline{\lim_{n\to\infty}} x_n = -\overline{\lim_{n\to\infty}}(-x_n)$.

Example. Consider the sequence $\{1, -1, 1, -1, \ldots\}$. It does not have a limit (as defined below) but does have limit-supremum $(= 1)$ and limit-infimum $(= -1)$.

For a sequence which does admit a limit, limit-supremum and limit-infimum are equal (try to prove it by yourself, or see the proof of Theorem A.4 below). For instance, $x_n = \dfrac{1}{n}$ has a limit $(= 0)$ which is both its lim-sup and its lim-inf.

(A.4) Limit of a sequence and Cauchy's criterion.– Recall that we say

$$\boxed{\ell = \lim_{n\to\infty} x_n}$$

if for every $\varepsilon > 0$, almost all of $x_n's$ are in the interval $(\ell - \varepsilon, \ell + \varepsilon)$. That is to say, there is some number $N > 0$ such that $|x_n - \ell| < \varepsilon$ for every $n \geq N$. If a sequence admits a limit, then the limit is necessarily unique.

To see this, assume ℓ_1 and ℓ_2 are two limits of the same sequence $\{x_n\}$, and $\ell_1 \neq \ell_2$. Let $d = |\ell_1 - \ell_2| > 0$ and pick $\varepsilon < \frac{d}{2}$. By their definitions, there are numbers $N_1, N_2 > 0$, so that every x_n with $n \geq N_1$ is in the interval $(\ell_1 - \varepsilon, \ell_1 + \varepsilon)$ and every x_m with $m \geq N_2$ is in the interval $(\ell_2 - \varepsilon, \ell_2 + \varepsilon)$. This is a contradiction, since these two intervals have empty intersection!

The following definition was recalled in Lecture 21, §21.1, page 2, as *Cauchy's criterion for convergence.*

Definition. A sequence $\{x_n\}_{n=0}^{\infty}$ is said to be *convergent* if for every $\varepsilon > 0$ there is a number $N > 0$ so that

$$|x_n - x_m| < \varepsilon \text{ for every } n, m \geq N.$$

The following theorem is also due to Cauchy[2], and is considered to be *the most important and fundamental theorem of analysis.*

Theorem. *A sequence $\{x_n\}_{n=0}^{\infty}$ has a limit (necessarily unique) if, and only if it is convergent.*

Proof. Let us assume $\ell = \lim_{n\to\infty} x_n$ exists. Let us check that the sequence $\{x_n\}$ has to be convergent. So, we are given $\varepsilon > 0$. By definition of the limit (applied to the number $\varepsilon/2$) we obtain a number $N > 0$ so that

$$|x_n - \ell| < \frac{\varepsilon}{2} \text{ for every } n \geq N.$$

Then, for every $n, m \geq N$, we have:

$$|x_n - x_m| = |(x_n - \ell) - (x_m - \ell)| \leq |x_n - \ell| + |x_m - \ell| < \varepsilon,$$

which proves that $\{x_n\}$ meets Cauchy's criterion of being convergent.

For the converse, let us consider a convergent sequence $\{x_n\}_{n=0}^{\infty}$. We will first check that it is bounded. Take $\varepsilon = 1$ in the definition above. Thus, we have a number $N > 0$ so that $|x_n - x_m| < 1$ for every $n, m \geq N$. The lower and upper bound for the entire sequence can then be taken as:

$$L < \text{Min}\{x_0, \ldots, x_{N-1}, x_N - 1\} \text{ and } R > \text{Max}\{x_0, \ldots, x_{N-1}, x_N + 1\}.$$

Having established that $\{x_n\}$ is bounded, we apply Theorem (A.2) to obtain two cluster points:

$$\ell_1 = \varliminf_{n\to\infty} x_n, \text{ and } \ell_2 = \varlimsup_{n\to\infty} x_n.$$

Now we are going to show that $\ell_1 = \ell_2$. This common value then has to be the limit of the sequence, by the second property of $\varliminf, \varlimsup$ in §A.3.

The proof of $\ell_1 = \ell_2$ follows the same logic as in the proof of uniqueness of limit given above. Namely, we argue by contradiction: if $\ell_1 \neq \ell_2$, let $d = |\ell_1 - \ell_2| > 0$ and take $\varepsilon < d/4$. By the first property of $\varliminf$ and $\varlimsup$, for this choice of ε, there are infinitely many $x_n's$ in each of the interval $(\ell_1 - \varepsilon, \ell_1 + \varepsilon)$ and $(\ell_2 - \varepsilon, \ell_2 + \varepsilon)$. But,

$$a \in (\ell_1 - \varepsilon, \ell_1 + \varepsilon) \text{ and } b \in (\ell_2 - \varepsilon, \ell_2 + \varepsilon) \text{ implies that } |a - b| > d/2.$$

So, $\{x_n\}$ does not meet Cauchy's criterion, if we pick ε there to be $d/2$. This is a contradiction, and we are done.

□

(A.5) Application 1. Descending chain property.– Assume that we have (bounded) closed intervals: $I_n = [a_n, b_n]$, $n \geq 0$ which form a descending chain:

$$I_0 \supset I_1 \supset I_2 \supset \cdots$$

[2] Augustin-Louis Cauchy (1789-1857) *Analyse Algébrique 1821.*

In addition, assume that $\lim_{n\to\infty} \text{length}(I_n) = 0$.

Thus, $a_0 \le a_1 \le a_2 \le \cdots < \cdots \le b_2 \le b_1 \le b_0$, and $(b_n - a_n) \to 0$, as $n \to \infty$.

Then:

$$a = \sup(\{a_0, a_1, a_2, \ldots\}) \le b = \inf(\{b_0, b_1, b_2, \ldots\}).$$

Now it is easy to see that $[a,b] = \cap_{n=0}^{\infty} I_n$. As the lengths of I_n are assumed to go to 0, we conclude that $a = b$. Therefore, we arrive at the following *descending chain property of bounded closed intervals*:

The intersection of a descending chain of bounded closed intervals, whose lengths approach 0*, consists of a single element.*

Remark. Some mathematicians take this property as the basic axiom of $\mathbb{R}$. This property is false for open intervals. For instance, the intersection of the following descending chain of bounded open intervals is empty:

$$(0,1) \supset (0,1/2) \supset (0,1/3) \supset \cdots$$

(A.6) Application 2. Absolute max/min.– We can now prove that every continuous function on a closed, bounded interval attains its maximum and minimum value.

Theorem. *Let $f : [a,b] \to \mathbb{R}$ be a continuous function. Then, there exists $M \in \mathbb{R}$ such that $f(x) \le M$ for every $x \in [a,b]$. In addition, there exists $c \in [a,b]$ such that $f(c) = M$.*

Similarly, there exists $m \in \mathbb{R}$ such that $f(x) \ge m$ for every $x \in [a,b]$. Additionally, $m = f(d)$ for some $d \in [a,b]$.

Proof. First of all, we have to show that the image of f is a bounded set. Namely, let $A = \{f(x) : x \in [a,b]\}$. Let us prove that there is some number $R \in \mathbb{R}$ such that $y < R$ for every $y \in A$. The same logic works to show the existence of some $L \in \mathbb{R}$ such that $L < y$ for every $y \in A$.

The proof is by contradiction (again). Assume there is no upper bound to the set A. Then for every positive integer $n \in \mathbb{Z}_{\ge 0}$ there must be some $x_n \in [a,b]$ so that $f(x_n) > n$. This way, we obtain a sequence of numbers $\{x_n\}_{n=0}^{\infty} \subset [a,b]$. By Bolzano–Weierstrass theorem A.2, this sequence has a cluster point, let us call that cluster point x^*. Take $y^* = f(x^*) \in \mathbb{R}$. As f was assumed to be continuous, for $\varepsilon = 1$, there must be some $\delta > 0$ that makes the following statement true:

$$\text{For every } x \text{ such that } 0 < |x - x^*| < \delta, \text{ we have } |f(x) - f(x^*)| < 1.$$

Meaning, the interval $(x^* - \delta, x^* + \delta)$, under our function f, lands inside $(y^* - 1, y^* + 1)$. But x^* was a cluster point of $\{x_n\}$, so the interval $(x^* - \delta, x^* + \delta)$ contains infinitely many of $x_n's$. This is a contradiction, since $f(x_n) > n$ has to eventually leave the interval $(y^* - 1, y^* + 1)$.

Having established that $A \subset \mathbb{R}$ is bounded, let $M = \sup(A)$ and $m = \inf(A)$. It remains to show that $f(c) = M$ for some $c \in [a,b]$ (the proof for m is verbatim).

For each integer $n \geq 1$, $M - 1/n$ is strictly smaller than M. By property of sup there must be some $c_n \in [a,b]$ so that $M - \frac{1}{n} < f(c_n) \leq M$. Again, we have found a sequence $\{c_n\}_{n=1}^{\infty} \subset [a,b]$ which must have a cluster point, by Theorem A.2. Let c be a cluster point of this sequence. Then, by continuity of f, $f(c) = M$.

□

(A.7) Heine–Borel theorem.– [3] Let $[a,b] \subset \mathbb{R}$ be a closed, bounded interval. Assume that there is a collection of open intervals $\{I_\ell\}_{\ell \in \Lambda}$ (here Λ is any set used to index $I_\ell's$) such that

$$[a,b] \subset \bigcup_{\ell \in \Lambda} I_\ell.$$

Theorem. *There exists a finite subset* $\{\ell_1, \ell_2, \ldots, \ell_n\} \subset \Lambda$ *such that*

$$[a,b] \subset I_{\ell_1} \cup I_{\ell_2} \cup \cdots \cup I_{\ell_n}.$$

Remark. This theorem is often stated as: *every open cover of a closed, bounded interval has a finite subcover.*

In general topological space X, a subset $K \subset X$ is said to be *compact* if every open cover of K has a finite subcover. This generalization is clearly inspired by Heine–Borel theorem. Thus, for $X = \mathbb{R}$ (or $\mathbb{C}$), *compact = closed and bounded.*

Proof. Let $S = \{t \in [a,b] : [a,t]$ is contained in a finite union of intervals from $\{I_\ell\}_{\ell \in \Lambda}\}$.

This set is non–empty, since $a \in S$. It is also an interval: meaning, if $t \in S$ and $s < t$, then $s \in S$. Therefore, it must be of the form $S = [a,T]$. We claim that $T = b$, which will prove the theorem. So, (again arguing by contradiction) let us assume that $T < b$. By construction of S, we can cover $[a,T]$ by finitely many intervals from the given collection $\{I_\ell\}_{\ell \in \Lambda}$:

$$[a,T] \subset I_{k_1} \cup I_{k_2} \cup \cdots \cup I_{k_m}.$$

Also, $T \in [a,b] \subset \bigcup_{\ell \in \Lambda} I_\ell$, meaning there is an open interval, say I_{ℓ_0} containing T. But that means (by definition of open intervals) that $(T - \varepsilon, T + \varepsilon) \subset I_{\ell_0}$ for some $\varepsilon > 0$. Therefore, we can cover $[a, T + \varepsilon/2]$ by finitely many intervals from the given collection $\{I_\ell\}_{\ell \in \Lambda}$:

$$[a, T + \varepsilon/2] \subset I_{k_1} \cup I_{k_2} \cup \cdots \cup I_{k_m} \cup I_{\ell_0}.$$

This contradicts the fact that $S = [a,T]$. Hence, $T = b$ and the theorem is proved. □

[3] Heinrich Eduard Heine (1821-1881). Émile Borel (1871-1956). The proof given here is due to Henri Lebasgue (1875-1941)