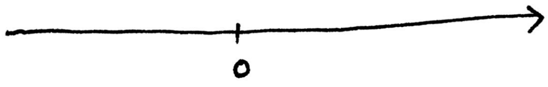
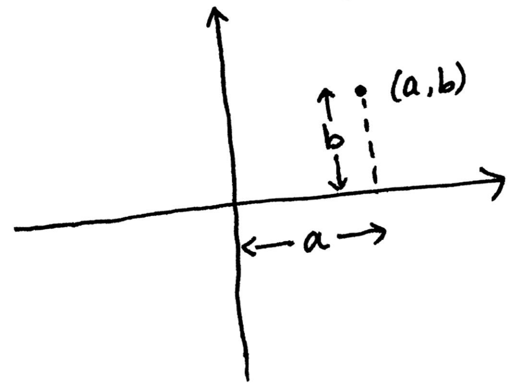
Lecture 0

(0.0) A complex number is defined to be a pair of real numbers. It is denoted by $z = a + bi$ where, a, b are real numbers.

iota.



Real line



Complex plane

Thus, a complex number can be viewed as a point in the two dimensional plane.

(0.1) Algebraic Operations.

(i) Addition of complex numbers is componentwise.

If $z_1 = a_1 + b_1 i$, $z_2 = a_2 + b_2 i$ then $z_1 + z_2 = (a_1 + a_2) + (b_1 + b_2) i$

(ii) Multiplication of complex numbers is given by:

If $z_1 = a_1 + b_1 i$ and $z_2 = a_2 + b_2 i$ then

$$\boxed{z_1 z_2 = (a_1 a_2 - b_1 b_2) + (a_1 b_2 + a_2 b_1) i}$$

Note: As a special case, when $z_1 = z_2 = i$ (that is, $a_1 = a_2 = 0$ and $b_1 = b_2 = 1$), we get $\boxed{i^2 = -1}$. In fact, we do not have to memorize the formula for multiplication since it follows from [(a) multiplication distributes over addition; and (b) $i^2 = -1$]

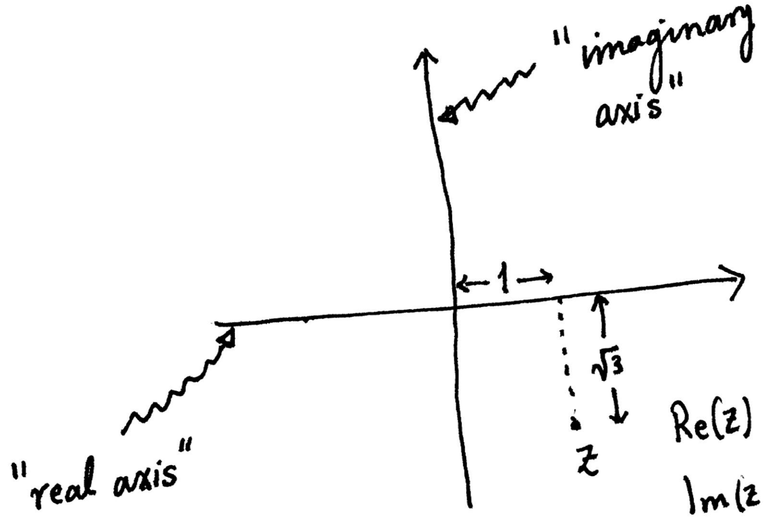
e.g. $z_1 = 1 + 3i$, $z_2 = 2 - 2i$.

$$\begin{aligned} z_1 \cdot z_2 &= (1+3i)\cdot(2-2i) \\ &= 1\cdot 2 + 1\cdot(-2i) + (3i)\cdot 2 + (3i)\cdot(-2i) \\ &= 2 - 2i + 6i - 6 i^2 \qquad \boxed{i^2 = -1} \\ &= 2 + 4i - 6(-1) \\ &= 8 + 4i. \end{aligned}$$

(0.2) For a complex number $z = a + bi$; a is called the <u>real part of z</u>, denoted by <u>$Re(z)$</u>, and b is called <u>the imaginary part of z</u>, denoted by $\mathrm{Im}(z)$.

e.g.

$z = 1 - \sqrt{3}\, i$



$\mathrm{Re}(z) = 1$

$\mathrm{Im}(z) = -\sqrt{3}$

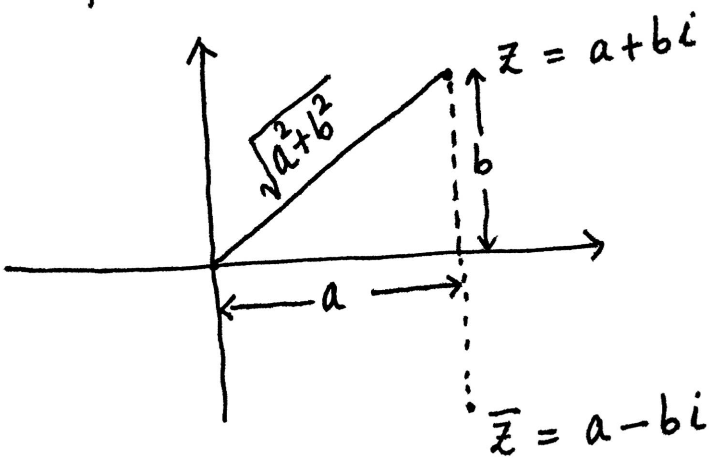
(0.3) Modulus, conjugate and inverse.

For $z = a+bi$, a complex number, we define

$|z| := \sqrt{a^2 + b^2}$ called <u>modulus of z</u>.

$\bar{z} := a - bi$ called <u>conjugate of z</u>

Note: $|z|$ is nothing but the distance between the points $(0,0)$ (i.e., origin) and (a,b) (Pythagorus' theorem).



($|z|$ is always a <u>non-negative real</u> number)

We also observe that $|z| = 0$ if and only if $z = 0$.

Let $z = a + bi$ be a non-zero complex number. Then,

$$z \cdot \overline{z} = (a+bi)(a-bi)$$

$$= a^2 + b^2 = |z|^2$$ is a ~~non~~- positive real number.

So, we can divide both sides by it.

$$\Rightarrow \quad z \cdot \left(\frac{\overline{z}}{|z|^2} \right) = 1$$

Thus, $z^{-1} = \dfrac{1}{z} = \dfrac{\overline{z}}{|z|^2}$.

(inverse of z) (just another notation)

e.g. Compute the real and imaginary parts of

$$\frac{1-2i}{2+3i}.$$

<u>Solution</u>.

$$\frac{1-2i}{2+3i} = \frac{1-2i}{2+3i} \cdot \frac{2-3i}{2-3i}$$

$$= \frac{(2-6) + (-4-3)i}{2^2+3^2} = \frac{-4-7i}{13}$$

Hence, $\mathrm{Re}\left(\dfrac{1-2i}{2+3i}\right) = \dfrac{-4}{13}$; $\mathrm{Im}\left(\dfrac{1-2i}{2+3i}\right) = \dfrac{-7}{13}$.

(0.4) Notation and short-hand.

Recall that $\mathbb{R}$ = set of real numbers. So, instead of writing "a is a real number", we write "$a \in \mathbb{R}$".

Similarly we let <u>$\mathbb{C}$ = set of complex numbers.</u>

Note : as a set, there is no difference between $\mathbb{C}$ and $\mathbb{R}^2$ (two-dimensional real vector space from Calculus). This new notation signifies the <u>multiplication</u> operation being defined above.

Properties of the algebraic operations on $\mathbb{C}$ (these are all obvious)

(i) $$z_1 + z_2 = z_2 + z_1$$
$$z_1 z_2 = z_2 z_1 \quad \text{for } z_1, z_2 \in \mathbb{C}.$$

[addition and multiplication are commutative.]

(ii) $$z_1 + (z_2 + z_3) = (z_1 + z_2) + z_3 \;;\; z_1(z_2 + z_3) = z_1 z_2 + z_1 z_3$$
$$z_1 (z_2 z_3) = (z_1 z_2) z_3 \quad \text{for all } z_1, z_2, z_3 \in \mathbb{C}$$

[addition and multiplication are ~~distributive~~ associative.]
multiplication <u>distributes</u> over addition.

(iii) $$0 + z = z \;;\quad 0 \cdot z = 0 \qquad \forall\, z \in \mathbb{C}.$$
$$1 \cdot z = z$$

symbol for "for every".

Finally, as we saw in § 0.3 above:

(iv) If $z \in \mathbb{C}$; $z \neq 0$, then $w = \dfrac{\bar{z}}{|z|^2} \in \mathbb{C}$ is such that $zw = 1$.

[every non-zero complex number can be inverted.]

Remark. Properties (i) – (iv) also hold for $\mathbb{R}$; and even for

$\mathbb{Q}$ = set of rational numbers.

$= \left\{ \dfrac{a}{b} \text{ where } a, b \text{ are integers } ; b \neq 0 \right\}$

(0.5) Some historical remarks*

(i) The earliest use of "$\sqrt{-1}$" is often attributed to Rafael Bombelli, around 1527.
(Italian school of Cordona)

(ii) The term "imaginary numbers" was coined by René Descartes, around 1620's.
(French philosopher)

(iii) The term "complex numbers" was used by Gauss (1822). He also used the symbol "iota" which replaced $\sqrt{-1}$ (used by Euler 1707 - 1783).

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* this will not be in the tests.