Lecture 1

(1.0) Recall: last time we defined complex numbers, and introduced the main operations on the set of complex numbers.

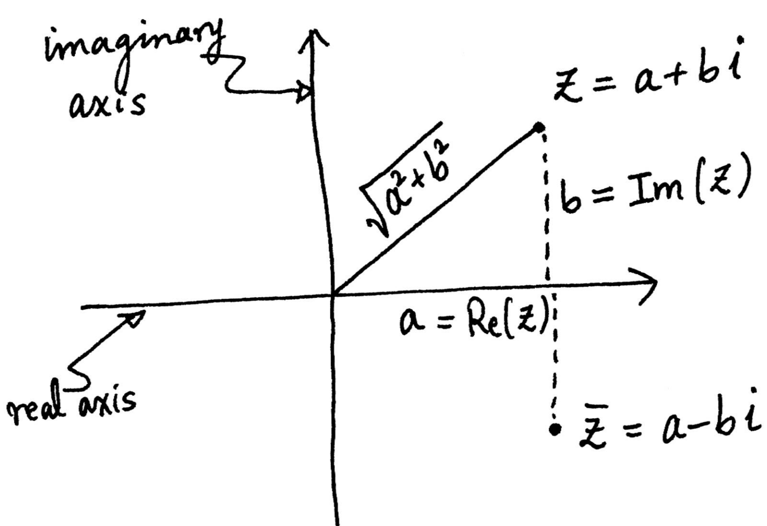
$\mathbb{C}$ = set of complex numbers
= set of pairs of real numbers ($\mathbb{R}$ = set of real numbers)
= $\{a+bi$ where $a, b \in \mathbb{R}\}$

- Addition: $(a_1+b_1 i) + (a_2+b_2 i) = (a_1+a_2) + (b_1+b_2) i$
- Multiplication (distributes over $+$; and $\boxed{i^2=-1}$):

$$(a_1+b_1 i)(a_2+b_2 i) = (a_1 a_2 - b_1 b_2) + (a_1 b_2 + a_2 b_1) i$$

- Modulus: $|a+bi| = \sqrt{a^2+b^2} \in \mathbb{R}_{\geq 0}$ (non-negative real number)

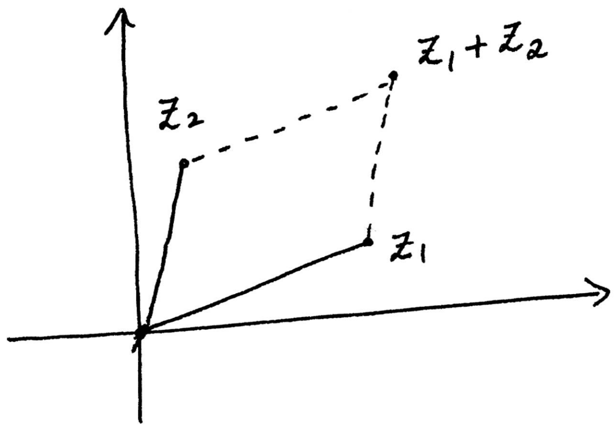
- Conjugate:
For $z = a+bi \in \mathbb{C}$, $\bar{z} = a-bi$ (called conjugate of z)



[Complex plane is just two dimensional real vector space] together with multiplication defined above.

(1.1) Geometrically, we can view addition of complex numbers (which, again, is same as vector addition from your previous calculus courses) by the familiar parallelogram rule :

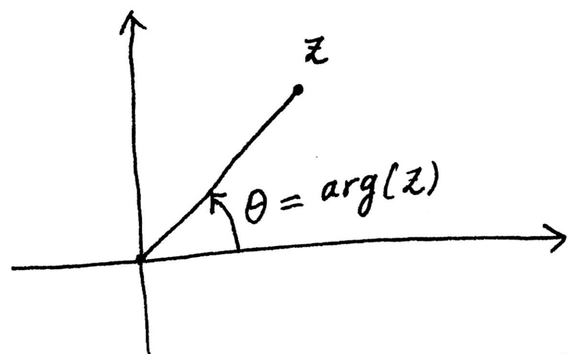
($0, z_1, z_2$ and $z_1 + z_2$ form vertices of a parallelogram).



(1.2) Argument (or phase) of a non-zero complex number :

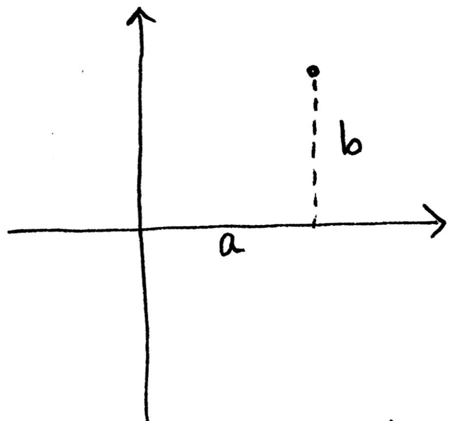
Let $z \in \mathbb{C}$, $z \neq 0$.

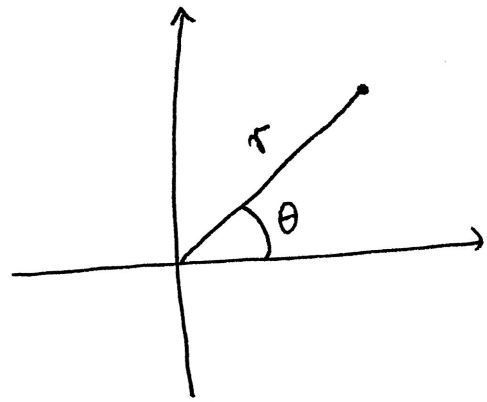
Argument of z, often denoted by $\arg(z)$, is the counter-clockwise angle formed by the line joining 0 and z, with the positive real axis (see figure to the right)



$|z|$ and $\arg(z)$ completely specify the (non-zero) complex number z.

Note: This is nothing but the <u>polar coordinates</u> we encountered earlier in a calculus course. Let us review the transformation between Cartesian and polar coordinates.





Cartesian Coordinates

$z = a + bi$

- $a = r\cos(\theta)$
- $b = r\sin(\theta)$

Polar Coordinates

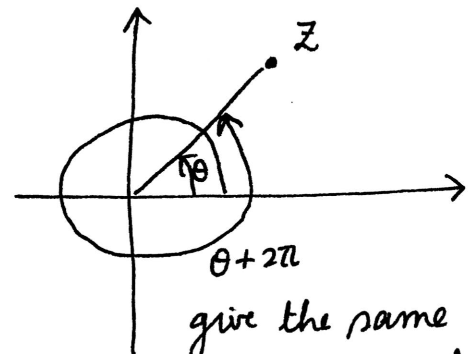
- $r = \sqrt{a^2+b^2} = |z|$
- θ is determined by:

$$\begin{cases} \cos(\theta) = \dfrac{a}{\sqrt{a^2+b^2}} \\ \sin(\theta) = \dfrac{b}{\sqrt{a^2+b^2}} \end{cases}$$

(NOT defined when $z = 0$)

<u>Warning</u>: Since $\cos(\theta + 2\pi) = \cos(\theta)$, if we know
$\sin(\theta + 2\pi) = \sin(\theta)$

cartesian coordinates of z (i.e., real & imaginary parts of z), its argument (or phase) is only determined up to an integer multiple of 2π.



give the same complex number!

e.g.

$$-1 = \cos(\pi) + \sin(\pi) i$$
$$= \cos(3\pi) + \sin(3\pi) i$$
$$= \cdots$$

To resolve this ambiguity, we (in this course) will agree to always take $\arg(z)$ in the interval:

$$-\pi < \arg(z) \leq \pi$$

[written as: $\arg(z) \in (-\pi, \pi]$]

$-\pi$ is not included π is included

(1.3) Examples

(i) $z = 1 + i$. $|z| = \sqrt{1^2 + 1^2} = \sqrt{2}$

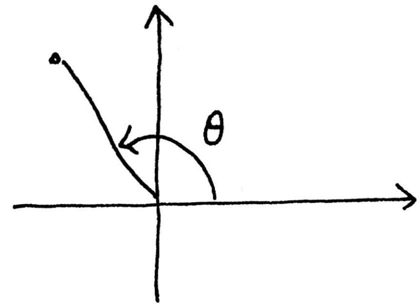
$\theta = \arg(z)$ is such that $\cos(\theta) = \dfrac{\mathrm{Re}(z)}{|z|} = \dfrac{1}{\sqrt{2}}$

$$\sin(\theta) = \frac{\mathrm{Im}(z)}{|z|} = \frac{1}{\sqrt{2}}.$$

So, $\theta = \dfrac{\pi}{4}$. That is,

$$z = 1 + i = \sqrt{2}\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} i\right) = \sqrt{2}\left(\cos\left(\frac{\pi}{4}\right) + \sin\left(\frac{\pi}{4}\right) i\right)$$

(ii) $z = -1 + \sqrt{3}\, i$



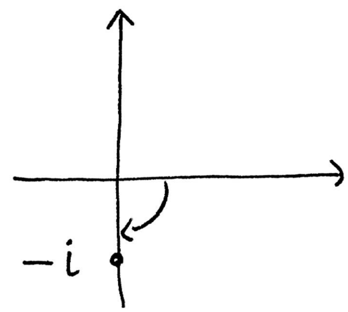
$$|z| = \sqrt{(-1)^2 + (\sqrt{3})^2} = \sqrt{4} = 2$$

$$\cos(\theta) = \frac{Re(z)}{|z|} = \frac{-1}{\sqrt{2}} \quad \text{and} \quad \sin(\theta) = \frac{Im(z)}{|z|} = \frac{\sqrt{3}}{2}$$

Thus $\theta = \pi - \frac{\pi}{3} = \frac{2\pi}{3} \quad \left(\in (-\pi, \pi] \right)$

(iii) $z = -i$

$$\arg(z) = \frac{-\pi}{2}.$$



(1.4) Summarizing: a non-zero complex number z is also written as

$$z = a + bi = r\left(\cos(\theta) + \sin(\theta)\, i\right)$$

$a = Re(z)$ $\qquad$ $r = |z|$

$b = Im(z)$ $\qquad$ $\theta = \arg(z)$

(1.5) Multiplication in polar coordinates:

Let $z_1 = r_1\left(\cos(\theta_1) + \sin(\theta_1)\, i\right)$

$z_2 = r_2\left(\cos(\theta_2) + \sin(\theta_2)\, i\right)$

be two non-zero complex numbers. Then:

$$z_1 \cdot z_2 = r_1 r_2 \left(\cos(\theta_1) + \sin(\theta_1) i\right)\left(\cos(\theta_2) + \sin(\theta_2) i\right)$$

$$= r_1 r_2 \Big(\left(\cos(\theta_1)\cos(\theta_2) - \sin(\theta_1)\sin(\theta_2)\right) + \left(\cos(\theta_1)\sin(\theta_2) + \sin(\theta_1)\cos(\theta_2)\right) i \Big)$$

Using the addition formulae from trigonometry :

$$\boxed{\begin{aligned} \cos(A+B) &= \cos(A)\cos(B) - \sin(A)\sin(B) \\ \sin(A+B) &= \sin(A)\cos(B) + \cos(A)\sin(B) \end{aligned}}$$

we get :

$$\boxed{z_1 z_2 = r_1 r_2 \left(\cos(\theta_1 + \theta_2) + \sin(\theta_1 + \theta_2) i \right)}.$$

Let us record this computation as :

$$|z_1 z_2| = |z_1| \cdot |z_2|$$

(modulus of product = product of moduli)

plural of modulus

$$\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$$

(argument of product = sum of arguments)

(recall the warning from pages 3-4 above : this equality is up to adding/subtracting integer multiples of 2π)

(1.6) Repeated application of the formulae from the last paragraph gives us:

$$\boxed{\left(\cos(\theta) + \sin(\theta)\, i\right)^n = \cos(n\theta) + \sin(n\theta)\, i}$$

for every $n = 1, 2, 3, \ldots$ (de Moivre's formula)

[Abraham de Moivre 1667–1754]

Example. Compute $(1+i)^8$.

$$1+i = \sqrt{2}\left(\cos\left(\frac{\pi}{4}\right) + \sin\left(\frac{\pi}{4}\right) i\right)$$ (see example from page 4)

$$\Rightarrow (1+i)^8 = (\sqrt{2})^8 \left(\cos\left(8\cdot\frac{\pi}{4}\right) + \sin\left(8\frac{\pi}{4}\right) i\right)$$

$$= 16\cdot\left(\cos(2\pi) + \sin(2\pi)\, i\right)$$

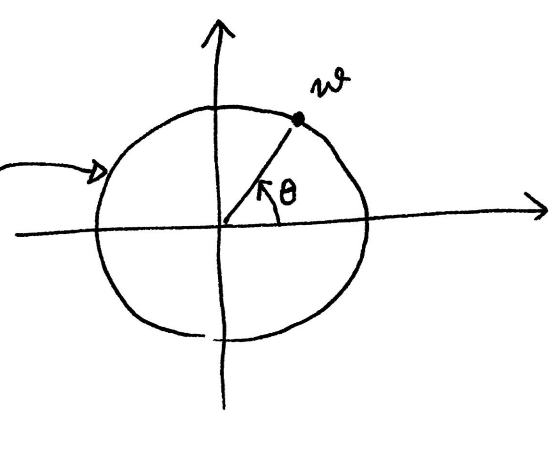
$$= 16.$$

(1.7) Geometric interpretation of multiplication.

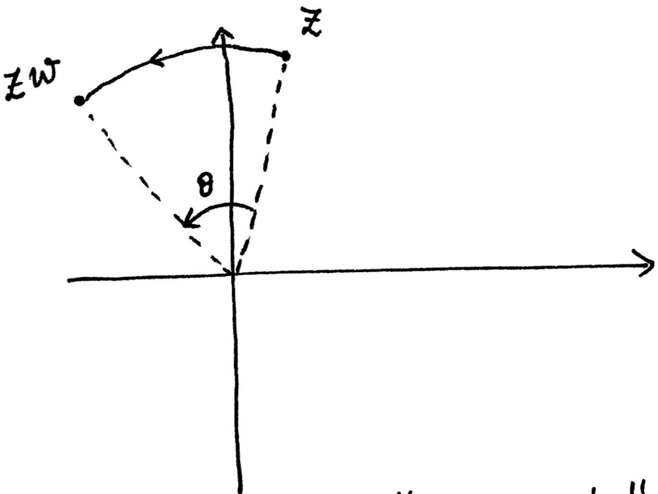
Let $w \in \mathbb{C}$ be a complex number of modulus 1.

So, $w = \cos(\theta) + \sin(\theta)\, i$.

circle of radius 1 centered at the origin = set of complex numbers of modulus 1



Again, by the computation we did in section (1.5) above, multiplying a complex number z by w is same as rotating z, counterclockwise, by an angle of θ $(= \arg(w))$.



(Multiplication of $z \in \mathbb{C}$ by w, where $|w| = 1$ and $\arg(w) = \theta$)

(1.8) Let us also record the "ambiguity" mentioned in section (1.2) above :

If $z_1 = r_1(\cos(\theta_1) + \sin(\theta_1)i)$ and $z_2 = r_2(\cos(\theta_2) + \sin(\theta_2)i)$ are two (non-zero) complex numbers, then :

($\mathbb{Z}$ = set of integers)

$z_1 = z_2$ if, and only if $r_1 = r_2$ and $\theta_1 - \theta_2 \in 2\pi\mathbb{Z}$

[read: θ_1 and θ_2 differ by an integer multiple of 2π]

(1.9) An example. Let us find all complex numbers $z \in \mathbb{C}$ such that $z^3 = 1$.

If $z = r(\cos(\theta) + \sin(\theta) i)$, then

$z^3 = r^3(\cos(3\theta) + \sin(3\theta) i)$. Our equation $z^3 = 1$

becomes : $r^3 = 1$ and 3θ is an integer multiple of 2π.

(hence $r = 1$)

- $3\theta = 0, 2\pi, 4\pi, 6\pi, \ldots$

$\theta = 0, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{6\pi}{3} = 2\pi, \frac{8\pi}{3} = 2\pi + \frac{2\pi}{3}$ and so on.

(give the same answer)

Therefore, $z^3 = 1 \Rightarrow$

$z = 1(\cos(0) + \sin(0) i) = 1$

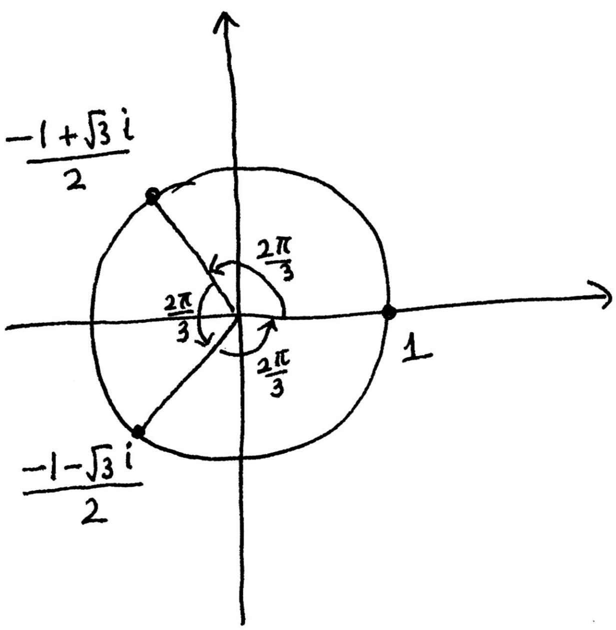
or

$z = 1\left(\cos\left(\frac{2\pi}{3}\right) + \sin\left(\frac{2\pi}{3}\right) i\right) = \frac{-1+\sqrt{3}\, i}{2}$

or

$z = 1\left(\cos\left(\frac{4\pi}{3}\right) + \sin\left(\frac{4\pi}{3}\right) i\right)$

$= \frac{-1-\sqrt{3}\, i}{2}$



(Three complex numbers that when cubed, give 1)