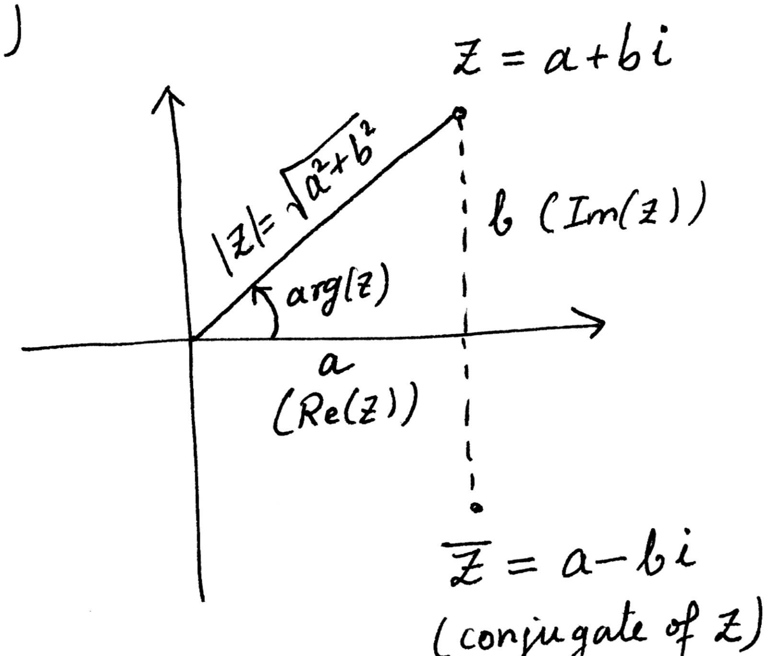
Lecture 2

(2.0) Recall: we defined complex numbers and introduced basic operations (addition, multiplication, modulus, argument and conjugate)



$$z = a + bi$$
$$= r(\cos(\theta) + \sin(\theta)i)$$
(assuming $z \neq 0$)

$$r = \sqrt{a^2+b^2} \; ; \quad \cos(\theta) = \frac{a}{\sqrt{a^2+b^2}}$$
$$\sin(\theta) = \frac{b}{\sqrt{a^2+b^2}}$$

Some properties we proved so far.

- $|z|^2 = z \cdot \overline{z}$
- $|z_1 z_2| = |z_1| \cdot |z_2|$
- $\overline{z_1 z_2} = \overline{z_1} \cdot \overline{z_2}$ (verify this)
- $\dfrac{z+\overline{z}}{2} = Re(z)$
- $\dfrac{z-\overline{z}}{2i} = Im(z)$
- $arg(z_1 z_2) = arg(z_1) + arg(z_2)$ (modulo 2π)
- $\left(r(\cos(\theta) + \sin(\theta)i)\right)^n = r^n(\cos(n\theta) + \sin(n\theta)i)$

(2.1) Last time we made the following observation:

If $z_1 = r_1\,(\cos(\theta_1) + \sin(\theta_1)\,i)$ and $z_2 = r_2(\cos(\theta_2) + \sin(\theta_2)\,i)$

are two non-zero complex numbers, then:

$\boxed{z_1 = z_2}$ is equivalent to $\boxed{\begin{array}{l} r_1 = r_2 \text{ , and} \\ \theta_1 - \theta_2 \in 2\pi\mathbb{Z} \end{array}}$

(2.2) n^{th} roots of a complex number.

Let n be a positive integer; and let $\alpha \in \mathbb{C}$ be a non-zero complex number. Let us find all complex numbers $z \in \mathbb{C}$ so that $\boxed{z^n = \alpha}$ (n^{th} roots of α).

Let us write α and z in their polar form:

$$\alpha = r\,(\cos(\varphi) + \sin(\varphi)\,i) \quad \leftarrow \text{given}$$

$$z = s\,(\cos(\theta) + \sin(\theta)\,i) \quad \leftarrow \text{unknown (to be computed)}$$

Then: $z^n = s^n(\cos(n\theta) + \sin(n\theta)\,i)$, and this turns our equation $\boxed{z^n = \alpha}$ into:

$$s^n(\cos(n\theta) + \sin(n\theta)) = r\,(\cos(\varphi) + \sin(\varphi)\,i).$$

By the observation made above, this means:

$s^n = r$ and $n\theta - \varphi$ is an integer multiple of 2π.

$\Rightarrow$ $\boxed{s = r^{1/n}}$ (remember: these are positive real numbers); and:

$$n\theta - \varphi = 0,\ 2\pi,\ 4\pi,\ 6\pi,\ \ldots$$

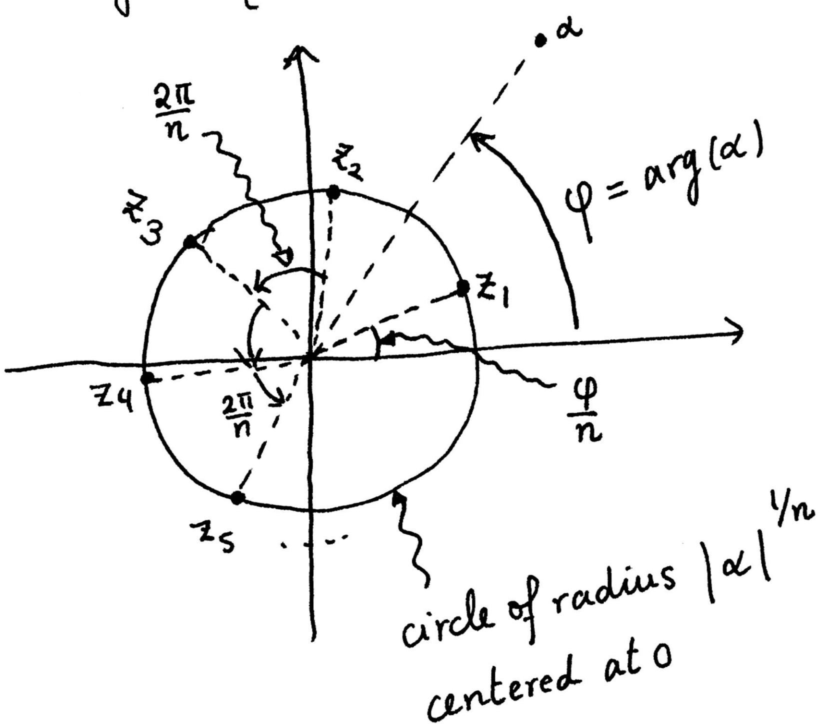
$$\Rightarrow\ n\theta = \varphi,\ \varphi + 2\pi,\ \varphi + 4\pi,\ \ldots$$

$$\theta = \frac{\varphi}{n},\ \frac{\varphi}{n} + \frac{2\pi}{n},\ \frac{\varphi}{n} + \frac{4\pi}{n},\ \ldots,\ \frac{\varphi}{n} + (n-1)\frac{2\pi}{n},$$

$$\frac{\varphi}{n} + n\cdot\frac{2\pi}{n},\ \frac{\varphi}{n} + (n+1)\frac{2\pi}{n},\ \ldots$$

will give the same z

Hence we get n distinct complex numbers whose n^{th} power is α, namely $\left\{ r^{1/n}\left(\cos\left(\varphi/n + k\cdot\frac{2\pi}{n}\right) + \sin\left(\frac{\varphi}{n} + k\cdot\frac{2\pi}{n}\right)i\right) : k = 0, 1, \ldots, n-1 \right\}$.



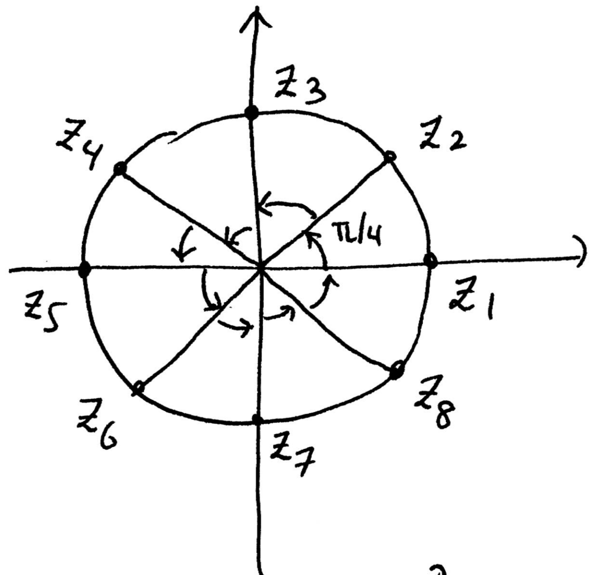
$\left[\begin{array}{c} n^{th} \text{ roots of } \alpha \in \mathbb{C} \\ \alpha \neq 0 \end{array}\right]$

(2.3) In conclusion: [Euler, 1707–1783.]

- n^{th} roots of α lie on the circle of radius $|\alpha|^{1/n}$ centered at 0.
- They form vertices of a regular n-gon (i.e. start from angle $\frac{\arg(\alpha)}{n}$, keep on rotating by $\frac{2\pi}{n}$).

e.g. $z^8 = 1$

($z_1 = 1$. Now keep rotating it by $\frac{2\pi}{8} = \frac{\pi}{4}$.)



(2.4) Example. Find all $z \in \mathbb{C}$ such that $z^3 = -8$.

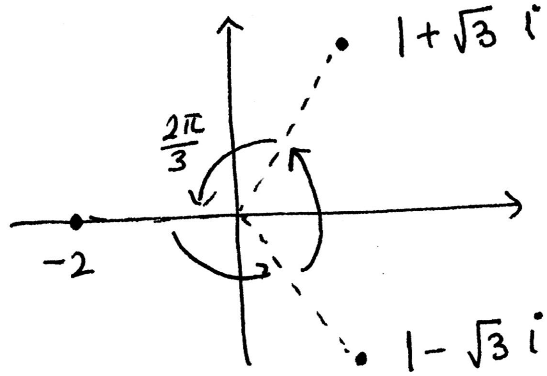
$$-8 = 8(\cos(\pi) + \sin(\pi)i)$$

$$z^3 = 8 \Rightarrow z = 2(\cos(\theta) + \sin(\theta)i) \text{ where}$$

$$\theta = \frac{\pi}{3} \text{ or } \frac{\pi}{3} + \frac{2\pi}{3} \text{ or } \frac{\pi}{3} + \frac{4\pi}{3}$$

$$\Rightarrow z = 2\left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) \text{ or } -2 \text{ or } 2\left(\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)$$

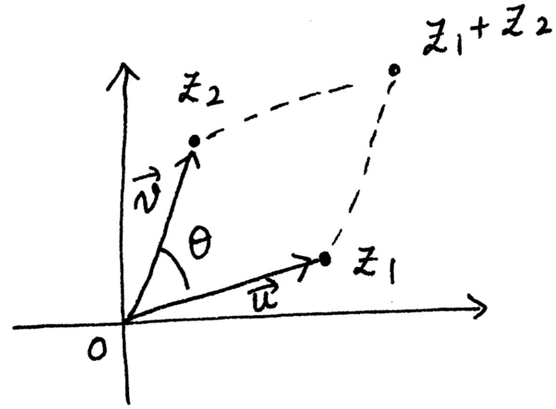
Solutions of $z^3 = -8$:



(2.5) Triangle inequality:

For $z_1, z_2 \in \mathbb{C}$: $\boxed{|z_1 + z_2| \leq |z_1| + |z_2|}$

Proof. (from Calculus).



If $\vec{u}$ = vector joining 0 to z_1
$\vec{v}$ = " " 0 to z_2
then $\vec{u}+\vec{v}$ = " " 0 to z_1+z_2.

$$|\vec{u}+\vec{v}|^2 = (\vec{u}+\vec{v})\cdot(\vec{u}+\vec{v}) \qquad \text{(dot product)}$$
$$= |\vec{u}|^2 + |\vec{v}|^2 + 2\,|\vec{u}|\cdot|\vec{v}|\cdot\cos(\theta)$$
$$\leq |\vec{u}|^2 + |\vec{v}|^2 + 2\,|\vec{u}|\cdot|\vec{v}| \qquad \left(\text{since } \cos(\theta) \leq 1\right)$$
$$= (|\vec{u}| + |\vec{v}|)^2$$

$\square$

(A more "complex-analytic" proof is outlined in Problem Sheet 1.)

(2.6) Using triangle inequality, we can also prove

$$\boxed{|z_1 - z_2| \geq \big|\, |z_1| - |z_2| \,\big|}$$

Proof: $|z_1| = |z_2 + z_1 - z_2|$

$\leq |z_2| + |z_1 - z_2|$ by triangle inequality

$\Rightarrow \quad |z_1| - |z_2| \leq |z_1 - z_2| \quad - (1)$

Switching the role of z_1 and z_2, we also get

$$|z_2| - |z_1| \leq |z_2 - z_1| = |z_1 - z_2| \quad - (2)$$

Combining (1) and (2), we get

$$\big|\, |z_1| - |z_2| \,\big| \leq |z_1 - z_2|$$

$\square$

(2.7) Some familiar curves in the two-dimensional plane, written in the language of complex numbers*:

- $|z - \alpha| = r$: circle of radius r, centered at α

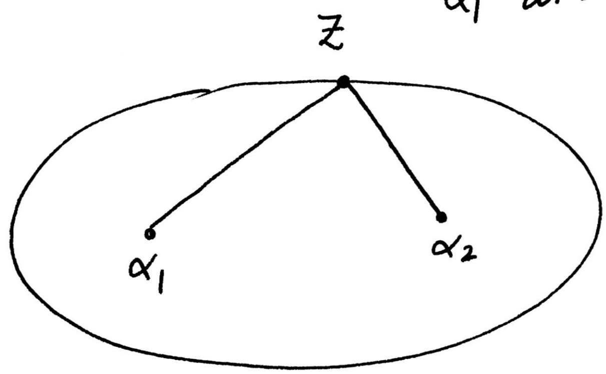
$\left(\underset{\text{fixed}}{\alpha \in \mathbb{C}}, \; r \in \mathbb{R}_{>0}\right)$

---

*optional

- Let $\alpha_1, \alpha_2 \in \mathbb{C}$ be fixed and let $r > |\alpha_1 - \alpha_2|$.

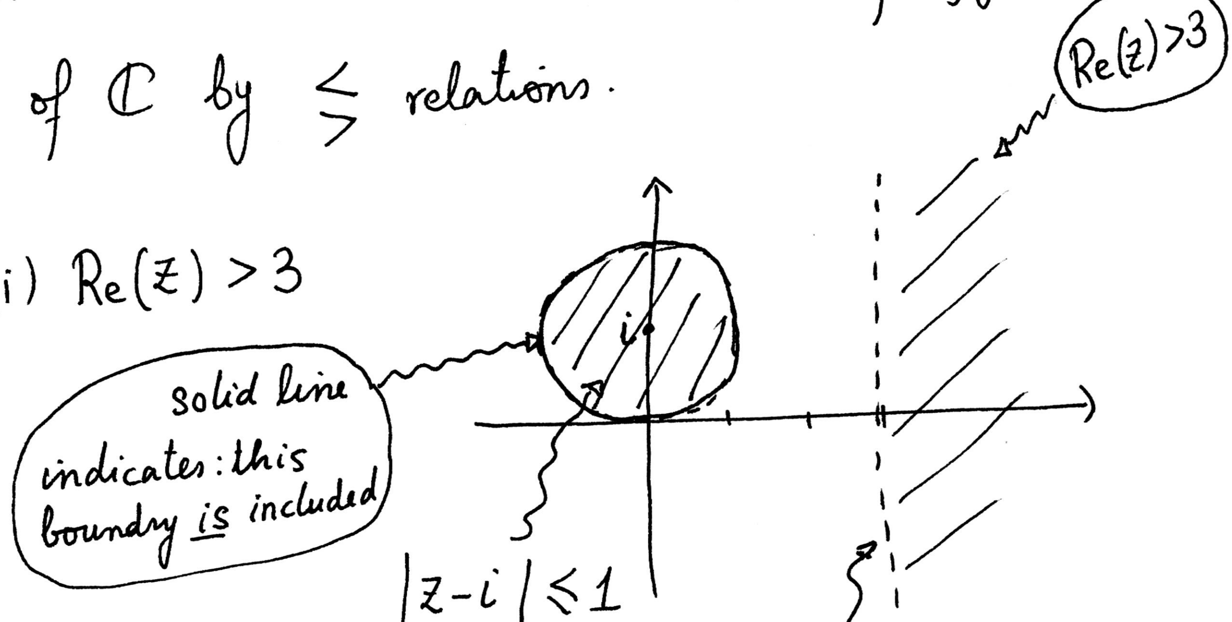
$|z-\alpha_1| + |z-\alpha_2| = r$: ellipse with foci at α_1 and α_2



[remember : $|z-w|$ = distance between z and w.]

(2.8) Later in this course, we will need to specify subsets of $\mathbb{C}$ by $\lesseqgtr$ relations.

Example : (i) $\mathrm{Re}(z) > 3$



(ii) $|z-i| \leq 1$

(closed disc of radius 1 centered at i)

dotted line means this line is not included.