Lecture 3

(3.0) We are going to study complex-valued functions of a complex variable $f : \mathbb{C} \longrightarrow \mathbb{C}$.

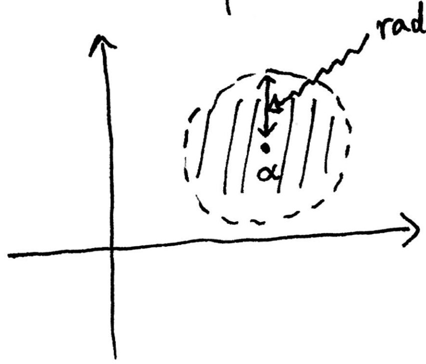
(input for f comes from $\mathbb{C}$) (f takes values in $\mathbb{C}$)

The domain of f, often (or always), will be an open and connected subset $\Omega \subseteq \mathbb{C}$. Let us begin by defining these, and other notions pertinent to subsets of the complex plane.

(3.1) Open and closed discs. Let $\alpha \in \mathbb{C}$ be a fixed complex number and let $r \in \mathbb{R}_{>0}$ be a positive real number.

$$D(\alpha; r) := \{ z \in \mathbb{C} \text{ such that } |z-\alpha| < r \}$$

$$= \text{(open) disc of radius } r \text{ centered at } \alpha$$



The closed disc of radius r, centred at α, is usually denoted by $\overline{D(\alpha; r)}$

$$\overline{D(\alpha;r)} = \{ z \in \mathbb{C} \text{ such that } |z-\alpha| \leq r \}.$$

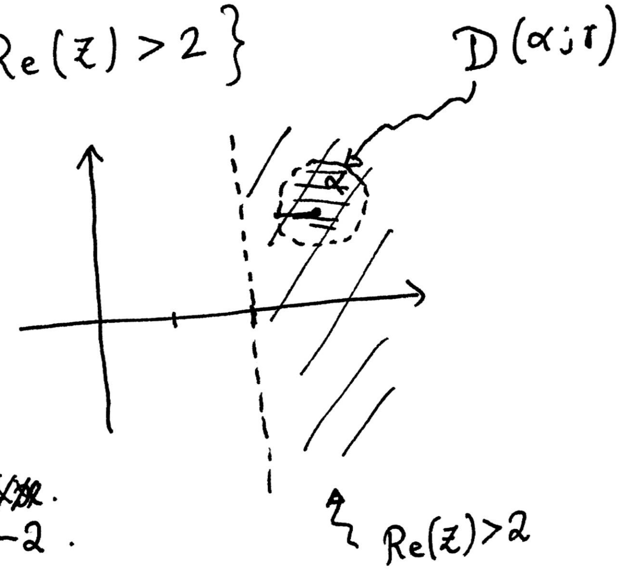
(3.2) Let $S \subseteq \mathbb{C}$ (a subset of the complex plane).

We say S is <u>open</u>, if for every $\alpha \in S$, we can find some $r \in \mathbb{R}_{>0}$ such that $D(\alpha; r) \subset S$.

A subset $T \subseteq \mathbb{C}$ is said to be <u>closed</u> if its complement $\underbrace{\mathbb{C} \setminus T}_{\text{set minus}}$ is open.

$$\Big[\mathbb{C} \setminus T = \{ z \in \mathbb{C} \underset{\text{such that}}{|} z \notin T \} \Big]$$

Examples. (i) $S = \{ z \in \mathbb{C} \text{ such that } \mathrm{Re}(z) > 2 \}$ is an open set.



[Proof. Let $\alpha \in S$. That is,

$\mathrm{Re}(\alpha) = x > 2$.

Choose $r \in \mathbb{R}$ so that $0 < r < x-2$.

Then : $D(\alpha; r) \subset S$. (easily justified - see picture)

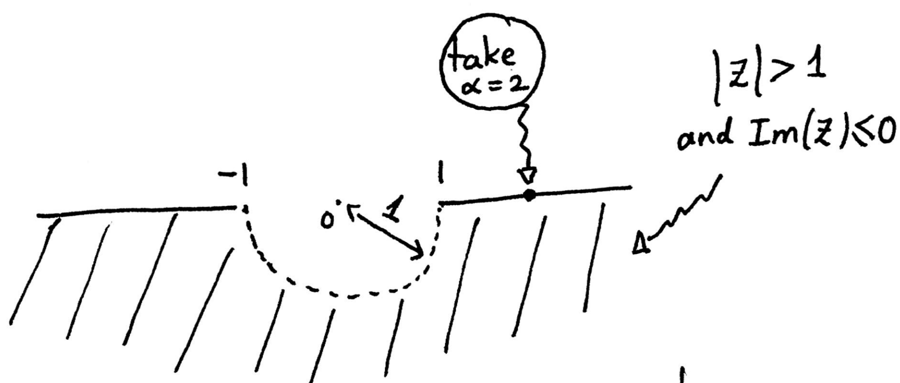
Hence, S is open. $\square$]

(ii) $S = \mathbb{C}$ is open.

(iii) Empty set is considered to be open. It is also closed, because its complement, $\mathbb{C}$, is open.

<u>Fact</u>: ϕ and $\mathbb{C}$ are the only subsets of $\mathbb{C}$ which are both <u>open and closed</u>.

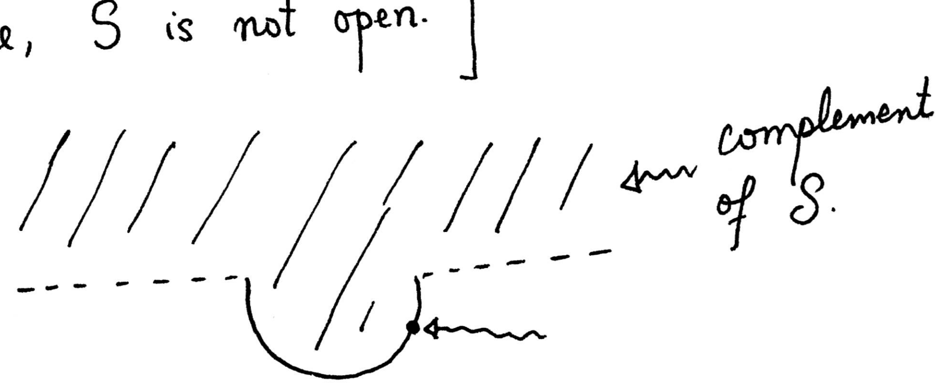
(iv) $S = \{ |z| > 1 \text{ and } \operatorname{Im}(z) \leq 0 \}$ is neither open, nor closed.



[Proof that S is not open:

- $2 \in S$
- for any $r \in \mathbb{R}_{>0}$, $D(2;r)$ contains a complex number of positive imaginary part, e.g. $2 + i\frac{r}{2}$. Thus $D(2;r) \not\subset S$. Hence, S is not open.]

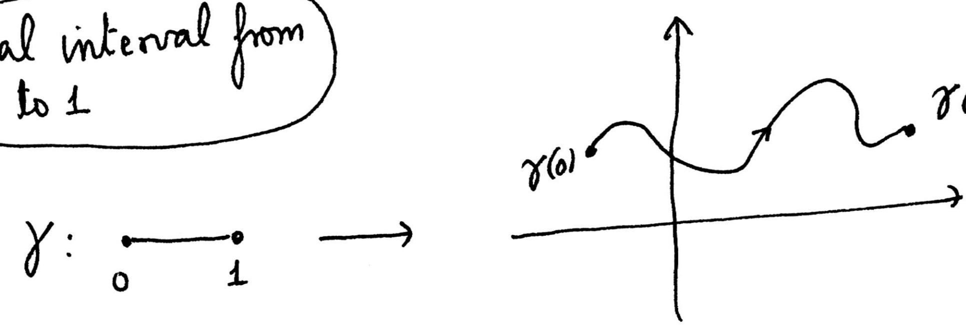
$T = \mathbb{C} \setminus S$:



is not open, by an argument similar to the one above. Hence, S is not closed.

(3.3) Connected-ness. A subset $S \subseteq \mathbb{C}$ is said to be <u>path-connected</u> if for any two given points $\alpha_1, \alpha_2 \in S$, we can find a path* starting at α_1, ending at α_2 while staying in S.

*: A <u>path</u> in the complex plane is a continuous function $\gamma : \underbrace{[0,1]}_{\text{real interval from 0 to 1}} \longrightarrow \mathbb{C}$



Thus, saying S is <u>path-connected</u> amounts to asking for a $\gamma : [0,1] \longrightarrow \mathbb{C}$; for every pair $\alpha_1, \alpha_2 \in \mathbb{C}$, so that $\gamma(0) = \alpha_1$, $\gamma(1) = \alpha_2$ and

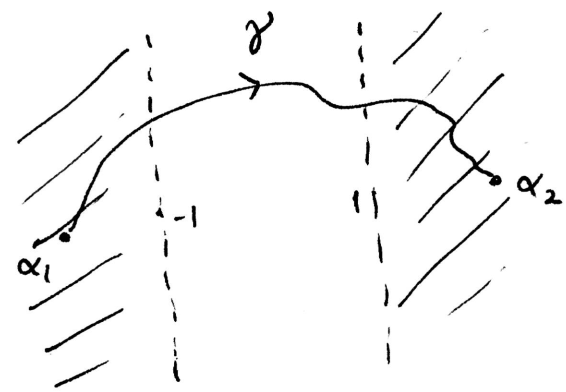
$$\gamma(t) \in S \text{ for all } 0 \leqslant t \leqslant 1.$$

<u>Remark</u>. In this course, we will use "connected" and "path-connected" interchangably. So, when we say $\boxed{S \text{ is connected}}$ we mean path-connected.

For general "topological spaces" these notions are different, but for complex plane we don't have to worry about that.

(3.4) The examples given in section (3.2) above are all connected.

$S = \{ |\mathrm{Re}(z)| > 1 \}$ is not connected.



[Proof*. Take $\alpha_1 = -2$
$\alpha_2 = 2$.

Let $\gamma : [0,1] \to \mathbb{C}$ be any path joining α_1 with α_2.

Consider $f(t) = \mathrm{Re}(\gamma(t))$.

$f : [0,1] \longrightarrow \mathbb{R}$ is a continuous function, with $f(0) = -2$ and $f(1) = 2$. A theorem (Bolzano) says that there must be some t_0, $0 < t_0 < 1$, such that $f(t_0) = 0$. Meaning, $\gamma(t_0) \notin S$. Hence, S is not connected. $\square$]

---

* optional.

(3.5) Accumulation points. Let $S \subseteq \mathbb{C}$ be a subset of $\mathbb{C}$.

Let $\alpha \in \mathbb{C}$. We say that α is <u>an accumulation point</u> <u>of S</u> if: for every $r \in \mathbb{R}_{>0}$, the <u>punctured disc</u> $\overset{\times}{D}(\alpha; r) := D(\alpha; r) \setminus \{\alpha\}$ intersects S non-trivially. In logical symbols, this statement is:

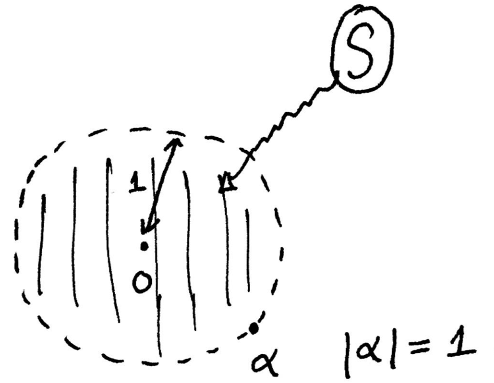
[Note: α may, or may not, be in S.]

$$\left[\forall\, r \in \mathbb{R}_{>0},\ \overset{\times}{D}(\alpha; r) \cap S \neq \phi. \right]$$

Examples. (i) $S = \{ |z| < 1 \}$

Let $\alpha \in \mathbb{C}$ be such that $|\alpha| = 1$.

Then α is an accumulation point of S.



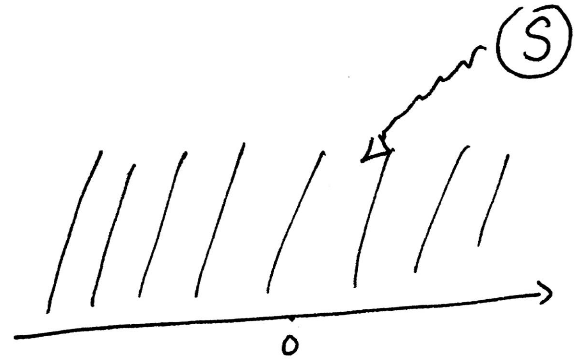
If $Acc(S)$ = set of accumulation points of S, then, in our example, $Acc(S) = \{ |z| \leq 1 \}$.

(ii) $S = \mathbb{Z} = \{0, \pm 1, \pm 2, \ldots\} \subset \mathbb{C}$.

$Acc(S) = \phi$. For instance, say $\alpha = 0$. Choose $0 < r < 1$. Then $\overset{\times}{D}(0; r) \cap S = \phi \Rightarrow 0$ is <u>not</u> an accumulation point.

(iii) $S = \{ \operatorname{Im}(z) \geq 0 \}$

$\operatorname{Acc}(S) = S$



(3.6) An alternate description of closed sets.

$$\boxed{\text{A set } S \subseteq \mathbb{C} \text{ is closed if and only if } \operatorname{Acc}(S) \subset S}.$$

Proof. ($\Rightarrow$) Assume first that S is closed. By definition, this means that its complement $A = \mathbb{C} \setminus S$ is open. That is, for every $\alpha \in A$, we can find some $r \in \mathbb{R}_{>0}$, such that $D(\alpha; r) \subset A$. That means $D(\alpha; r) \cap S = \phi$. We have, therefore, shown that

$\alpha \in A$ (i.e. $\alpha \notin S$) $\Rightarrow$ α is not an accumulation point of S.

which is same as saying $\boxed{\alpha \text{ is an accumulation point} \Rightarrow \alpha \in S}$.

i.e., $\operatorname{Acc}(S) \subset S$.

($\Leftarrow$) left as an exercise (convince yourself that the argument given above can be reversed - word-by-word!)

(3.7) A subset $S \subset \mathbb{C}$ is said to be <u>bounded</u> if there exists some real (positive) number $N \in \mathbb{R}_{>0}$ such that $D(0;N)$ contains S entirely.

$$[\, S \subset D(0;N) \,]$$

That is to say : $\boxed{\alpha \in S \Rightarrow |\alpha| < N}$

A <u>closed and bounded set</u> is also called <u>compact</u>.

- Example (i) of (3.2) is neither closed nor bounded.
- $\overline{D(0;r)} = \{|z| \leq r\}$ is both closed and bounded (i.e. compact).
- $\mathbb{Z} \subset \mathbb{C}$ is closed but not bounded.
- $D(0;1) = \{|z| < 1\}$ is open & bounded (NOT compact).