Lecture 4

(4.0) Recall : last time we defined <u>open subsets</u> of $\mathbb{C}$:

$S \subseteq \mathbb{C}$ is open if for every $\alpha \in S$, there exists some positive real number r, such that :

$$D(\alpha ; r) \subset S$$

open disc $\{ |z-\alpha| < r \}$ (centered at α, radius $= r$).

(4.1) Today we are going to begin our study of <u>complex-valued functions of a complex variable</u>.

Let $\Omega \subseteq \mathbb{C}$ be an open set.

$$\boxed{f : \Omega \longrightarrow \mathbb{C}}$$

($\mathbb{C}$ - valued function of one complex variable)

- f takes inputs from the subset $\Omega \subseteq \mathbb{C}$ (domain of f)
- For every $z \in \Omega$; $f(z)$ is a complex number.

Writing $z = x + yi$, we immediately see that f is
$(x, y \in \mathbb{R})$

completely specified by two real-valued functions of two real variables:

$$f(\underbrace{x+yi}) = \underbrace{u(x,y)}_{\mathrm{Re}(f(z))} + \underbrace{v(x,y)}_{\mathrm{Im}(f(z))} i$$

$z = x+yi \in \Omega$

e.g. $(\Omega = \mathbb{C})$ $f(x+yi) = (x^2+y^2) + (xy)i$

e.g. $(\Omega = \mathbb{C} \setminus \{0\})$ $f(x+yi) = \dfrac{x}{x^2+y^2} + \dfrac{y^2}{x^2+y^2} i$

[Pick your favourite (two) functions $u(x,y)$ and $v(x,y)$ from Calculus III $\rightsquigarrow$ $f(x+yi) = u(x,y) + v(x,y)i$.]

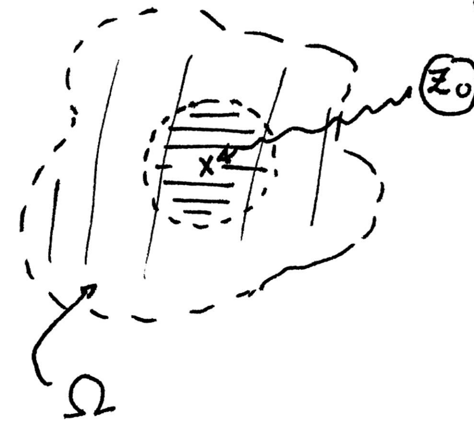
(4.2) Our very first task is going to be: definition of limits and continuity of a function of a complex variable.

Set up:
- $\Omega \subseteq \mathbb{C}$ open set
- $f : \Omega \to \mathbb{C}$
- $z_0 \in \mathbb{C}$ such that there is some positive real

number r for which $D^{\times}(z_0; r) \subset \Omega$

$\boxed{\text{punctured disc} : 0 < |z - z_0| < r}$

Meaning (in words) :



$\boxed{\text{$f$ may, or may not, be defined at z_0. But we want f to be defined in a punctured disc around z_0.}}$

e.g. $f(x+yi) = \dfrac{1}{(x-1)^2+y^2} + \dfrac{2x}{x^2+(y-3)^2}\, i$ $\Bigg\}$ meets the requirements set up above.

$\Omega = \mathbb{C} \setminus \{1, 3i\}$. $z_0 = 3i$

(4.3) $\boxed{\lim_{z \to z_0} f(z) = A}$ means the following :

For every real, positive number ε, we can find (again, real, positive) δ, such that :

$$0 < |z - z_0| < \delta \text{ implies } |f(z) - A| < \varepsilon .$$

[In logical symbols: $\forall\, \varepsilon > 0$, $\exists\, \delta > 0$ such that
$$0 < |z - z_0| < \delta \Rightarrow |f(z) - A| < \varepsilon .$$]

Definition: We say f is <u>continuous</u> at z_0, if

- $\lim_{z \to z_0} f(z)$ exists
- $z_0 \in \Omega$ (i.e. f is defined at z_0)
- $\lim_{z \to z_0} f(z) = f(z_0)$

(4.4) Understanding $\boxed{\lim_{z \to z_0} f(z) = A}$

(read: limit of $f(z)$, as z approaches z_0, is $A \in \mathbb{C}$.
Also written as: $\boxed{f(z) \to A \quad \text{as} \quad z \to z_0}$.)

In plain english, our definition says: we can make the value of $f(z)$ as close to A as we want, by choosing our inputs to lie in a small enough (punctured) disc around z_0.

Even this notion is nothing new. Let us write real and imaginary parts separately:

$$f(x+yi) = u(x,y) + v(x,y)i$$

$$z_0 = x_0 + y_0 i \quad ; \quad A = a + bi$$

Then : $\boxed{\lim_{z \to z_0} f(z) = A}$ is equivalent to $\boxed{\begin{array}{c} \lim_{(x,y)\to(x_0,y_0)} u(x,y) = a \\ \underline{\text{and}} \\ \lim_{(x,y)\to(x_0,y_0)} v(x,y) = b \end{array}}$

(These notions were introduced in Calculus III)

Proof*. This proof uses two properties of the limit :

(i) $\lim_{z \to z_0} f(z) = A \Rightarrow \lim_{z \to z_0} \overline{f(z)} = \overline{A}$ and conversely.

(clear from our definition since $|f(z) - A| = |\overline{f(z)} - \overline{A}|$.)

(ii) $\left.\begin{array}{l} \lim_{z\to z_0} g_1(z) = A_1 \\ \lim_{z \to z_0} g_2(z) = A_2 \end{array}\right\} \Rightarrow \lim_{z \to z_0} \left(g_1(z) + g_2(z)\right) = A_1 + A_2$

(use triangle inequality : let $\varepsilon > 0$ be given. By definition of the limit, we can find $\delta > 0$ so that

$$0 < |z - z_0| < \delta \Rightarrow \begin{array}{l} |g_1(z) - A_1| < \frac{\varepsilon}{2} \\ |g_2(z) - A_2| < \frac{\varepsilon}{2} \end{array}$$

triangle ineq.

This implies $|g_1(z) + g_2(z) - A_1 - A_2| \leq |g_1(z) - A_1| + |g_2(z) - A_2|$

$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$. □)

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* optional

Now, $\lim\limits_{(x,y)\to(x_0,y_0)} u(x,y) = \lim\limits_{z\to z_0} \operatorname{Re} f(z)$

$$= \lim_{z\to z_0} \frac{f(z)+\overline{f(z)}}{2} = \frac{A+\overline{A}}{2} = \operatorname{Re}(A) = a.$$

Similarly, with the $\operatorname{Im}(f(z)) = v(x,y)$ $\square$

(4.5) Examples. (just for a real-valued function of 2 real variables)

(i) $u(x,y) = \dfrac{xy}{x^2+y^2}$. $\lim\limits_{(x,y)\to(0,0)} u(x,y)$ does not exist.

- approach $(0,0)$ by setting $y=0$ and let $x\to 0^+$:

$$\lim_{x\to 0^+} u(x,0) = 0.$$

- approach $(0,0)$ by setting $x=y=t$ and let $t\to 0^+$:

$$\lim_{t\to 0^+} u(t,t) = \frac{1}{2}.$$

[If we can get two different answers by approaching $(0,0)$ in two different ways, we conclude that the limit does not exist.]

(ii) $u(x,y) = x+y$. $\lim\limits_{(x,y)\to(0,0)} u(x,y) = 0$.

Proof. Let $\varepsilon > 0$ be given. Pick δ : $0 < \delta < \frac{\varepsilon}{2}$.

Then for every (x,y) so that $\sqrt{x^2+y^2} < \delta$, we have:

$$|u(x,y) - 0| = |x+y| \underset{\substack{\uparrow \\ \text{triangle ineq.}}}{\leq} |x| + |y| \leq \sqrt{x^2+y^2} + \sqrt{x^2+y^2} < 2\delta < \varepsilon.$$

$\square$