Lecture 5

(5.0) Recall: last time we defined the notions of limit and continuity, for a function of a complex variable.

$$\boxed{\lim_{z \to z_0} f(z) = A}$$ means For every $\varepsilon > 0$, there exists $\delta > 0$ such that: $0 < |z - z_0| < \delta \Rightarrow |f(z) - A| < \varepsilon$

which in turn amounts to the idea encountered already in Calculus previously: if $\begin{cases} f(z) = u(x,y) + v(x,y)i & \left(\begin{matrix} x = Re(z) \\ y = Im(z) \end{matrix}\right) \\ z_0 = x_0 + y_0 i \\ A = a + bi \end{cases}$

then $\boxed{\lim_{z \to z_0} f(z) = A}$ is <u>equivalent</u> to $\lim_{(x,y) \to (x_0,y_0)} u(x,y) = a$; <u>and</u> $\lim_{(x,y) \to (x_0,y_0)} v(x,y) = b$.

(5.1) In the same spirit, the usual properties of limits hold:

(i) $\lim_{z \to z_0} (f_1(z) + f_2(z)) = \lim_{z \to z_0} f_1(z) + \lim_{z \to z_0} f_2(z)$ [if these two exist.]

(ii) $\lim_{z \to z_0} (c \cdot f(z)) = c \cdot \lim_{z \to z_0} f(z)$ ($c \in \mathbb{C}$ is any complex number). [if the limit exists]

(iii) $\lim_{z \to z_0} \frac{f(z)}{g(z)} = \frac{\lim_{z \to z_0} f(z)}{\lim_{z \to z_0} g(z)}$ [if the limits exist and $\lim_{z \to z_0} g(z) \neq 0$.]

(iv) $\lim_{z \to z_0} (f(z) \cdot g(z)) = \left(\lim_{z \to z_0} f(z)\right) \cdot \left(\lim_{z \to z_0} g(z)\right)$ [if both these limits exist]

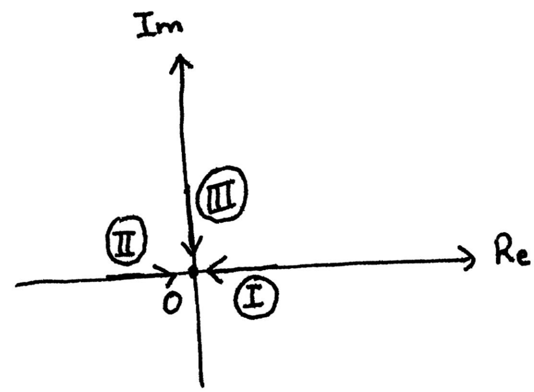
(5.2) Two more examples.

(1) Let $f(z) = \dfrac{z}{|z|}$ $\left(\begin{array}{c}\text{domain } \Omega = \mathbb{C} \sim \{0\} \\ z_0 = 0\end{array}\right)$

$\displaystyle\lim_{z \to 0} \frac{z}{|z|}$ does not exist. Note: $\dfrac{z}{|z|} = \dfrac{x}{\sqrt{x^2+y^2}} + \dfrac{y}{\sqrt{x^2+y^2}}\, i$

$(x = \mathrm{Re}(z),\ y = \mathrm{Im}(z))$.

Proof. $u(x,y) = \mathrm{Re}\left(\dfrac{z}{|z|}\right) = \dfrac{x}{\sqrt{x^2+y^2}}$



I. Approach $(0,0)$ by setting $y = 0$; $x \to 0^+$

$$\lim_{\substack{(x,y)\to(0,0)\\ \text{along I}}} u(x,y) = \lim_{x \to 0^+} \frac{x}{x} = 1.$$

II. Approach $(0,0)$ by setting $y = 0$; $x \to 0^-$

$$\lim_{\substack{(x,y)\to(0,0)\\ \text{along II}}} u(x,y) = -1.$$

$\square$

(Ex. along III, we get 0)

(2) Let $f(z) = z^3$ (domain $\Omega = \mathbb{C}$) $z_0 \in \mathbb{C}$ any complex number (fixed!)

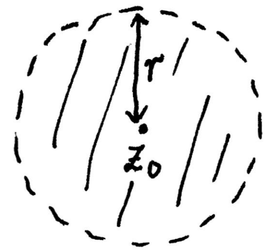
Prove : $\displaystyle\lim_{z \to z_0} z^3 = z_0^3$.

Proof* $|z^3 - z_0^3| = |(z - z_0)(z^2 + z z_0 + z_0^2)|$

$$= |z - z_0| \cdot |z^2 + z z_0 + z_0^2|$$

$$\leq |z - z_0| \cdot \left(|z|^2 + |z| \cdot |z_0| + |z_0|^2\right) \quad — (1)$$

*optional (see section (5.3) below).



Let $M > 0$ be large enough so that

$$|z_0| < \frac{M}{2} \quad (< M).$$

If $z \in D(z_0; r)$ then

$$\begin{aligned} |z| &= |z - z_0 + z_0| \\ &\leq |z - z_0| + |z_0| \\ &< r + |z_0| \end{aligned}$$

Then: for $z \in D(z_0; \frac{M}{2})$,

$$\begin{aligned} |z| < \frac{M}{2} + |z_0| &< \frac{M}{2} + \frac{M}{2} \\ &= M. \end{aligned}$$

and we get a bound:

$$|z|^2 + |z| \cdot |z_0| + |z_0|^2 < 3M^2.$$

So, let there be given $\varepsilon > 0$. Pick $\delta > 0$ so that $\delta < \frac{M}{2}$ and $\delta < \frac{\varepsilon}{3M^2}$. Then: for every $z \in D^*(z_0; \delta)$ (i.e. $0 < |z - z_0| < \delta$)

we get:

$$|z^3 - z_0^3| \leq |z - z_0| \cdot \left(|z|^2 + |z|\,|z_0| + |z_0|^2\right) \quad \text{(from (1) of last page)}$$

$$< \frac{\varepsilon}{3M^2} \cdot 3M^2 = \varepsilon.$$

$\square$

(5.3) It is much easier to prove that $\lim\limits_{z \to z_0} z = z_0$.

Using property (iv) of Section (5.1) above repeatedly, we conclude: $\boxed{\lim\limits_{z \to z_0} z^n = z_0^n}$ (for every $n = 1, 2, 3, \ldots$)

Using properties (i) and (ii) of (5.1) above, we get:

If $f(z) = a_n z^n + a_{n-1} z^{n-1} + \dots + a_1 z + a_0$ (polynomial function)

$(a_0, a_1, \dots, a_n \in \mathbb{C} \; ; \; n \in \mathbb{Z}_{\geq 0})$

Then $\lim\limits_{z \to z_0} f(z) = f(z_0)$

(in other words, polynomial functions are continuous <u>everywhere</u>)

Using property (iii), if $f(z)$ and $g(z)$ are polynomial functions, and $z_0 \in \mathbb{C}$ is so that $g(z_0) \neq 0$; then

$$\lim_{z \to z_0} \frac{f(z)}{g(z)} = \frac{f(z_0)}{g(z_0)}$$

(called rational function). (In other words, rational functions are continuous, <u>wherever defined</u>.)

e.g. $\displaystyle \lim_{z \to i} \frac{z^2 + 2}{z + 1} = \frac{i^2 + 2}{i + 1} = \frac{1}{1+i} = \frac{1-i}{2}$.

(5.4) $\mathbb{C}$ - differentiability.

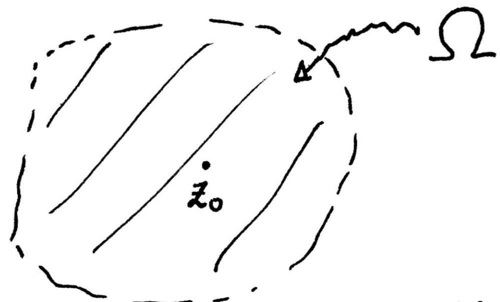
The concept of <u>complex differentiability</u> is going to be significantly different from (real) differentiability, even though the definition will be almost verbatim.

This concept was introduced by Cauchy in his famous text Cours d'analyse (1821)

[Augustin-Louis Cauchy 1789 – 1857, Paris, France]

Definition. Let $f : \Omega \to \mathbb{C}$ be a function (of complex variable). $\Omega \subseteq \mathbb{C}$ here is an open set. Let $z_0 \in \Omega$.

We say f is $\mathbb{C}$-differentiable at z_0



if

$$\lim_{\substack{h \to 0 \\ (h \in \mathbb{C})}} \frac{f(z_0+h) - f(z_0)}{h} \text{ exists.}$$

This is what is new

[often denoted by $f'(z_0)$ or $\left.\frac{df}{dz}\right|_{z=z_0}$.]

(5.5) This definition is clearly inspired by its counterpart from functions of real variable. Let us see what is so novel about it.

$f(z) = u(x,y) + v(x,y)i$.
$(x = Re(z),\ y = Im(z))$

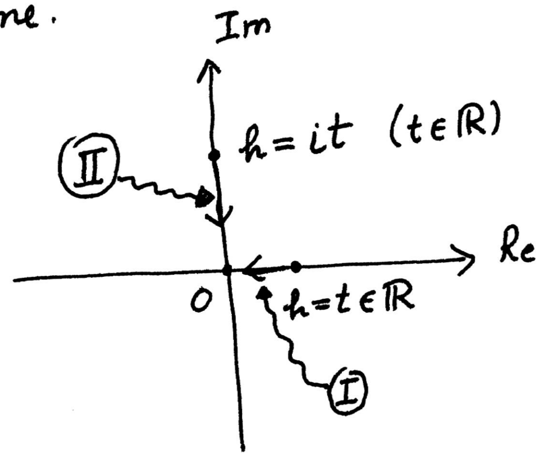
$z_0 = x_0 + y_0 i$.

Assume that the limit

$$\lim_{h \to 0} \frac{f(z_0+h) - f(z_0)}{h}$$

exists.

So, we can let $h \in \mathbb{C}$ approach 0 whichever way we please and the answer must be the same.



I : Take $h = t \in \mathbb{R}$ and let $t \to 0$. The limit in question becomes:

$$\lim_{\substack{t \to 0 \\ (t \in \mathbb{R})}} \frac{f(z_0+t) - f(z_0)}{t} = \lim_{t \to 0} \left[\frac{u(x_0+t, y_0) - u(x_0, y_0)}{t} + \frac{v(x_0+t, y_0) - v(x_0, y_0)}{t} i \right]$$

$$\underset{\text{(by definition of partial derivatives)}}{=} \left.\frac{\partial u}{\partial x}\right|_{(x_0,y_0)} + \left(\left.\frac{\partial v}{\partial x}\right|_{(x_0,y_0)} \right) i$$

$$= \boxed{u_x(x_0, y_0) + v_x(x_0, y_0)\, i} \qquad \text{(just another notation for partial derivatives).}$$

II : Take $h = it \quad (t \in \mathbb{R})$ and let $t \to 0$.

$$\lim_{\substack{t \to 0 \\ (t \in \mathbb{R})}} \frac{f(z_0+it) - f(z_0)}{it} = \frac{1}{i} \lim_{t \to 0} \left[\frac{u(x_0, y_0+t) - u(x_0, y_0)}{t} + \frac{v(x_0, y_0+t) - v(x_0, y_0)}{t} i \right]$$

$$= \frac{1}{i} \left(u_y(x_0, y_0) + v_y(x_0, y_0)\, i \right)$$

$$= \boxed{v_y(x_0, y_0) - i\, u_y(x_0, y_0).}$$

As these two answers are supposed to be the same, we get a non-trivial condition on the functions $u(x,y)$ and $v(x,y)$, known as Cauchy-Riemann* equations:

* Georg Friedrich Bernhard Riemann (1826–1866)

$$\boxed{\begin{aligned} u_x(x_0,y_0) &= v_y(x_0,y_0) \\ u_y(x_0,y_0) &= -v_x(x_0,y_0) \end{aligned}}$$

Cauchy-Riemann Equations

(5.6) The argument given above proves the following:

$$\boxed{\begin{array}{l} f(z) = u(x,y) + v(x,y)i \\ \quad (x = Re(z),\ y = Im(z)) \\ \text{is } \mathbb{C}\text{-differentiable} \end{array}} \implies \boxed{\begin{array}{l} \dfrac{\partial u}{\partial x}, \dfrac{\partial u}{\partial y}, \dfrac{\partial v}{\partial x}, \dfrac{\partial v}{\partial y} \text{ exist} \\ \dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y} \\ \dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x} \end{array}}$$

In the next lecture, we will prove the converse of this statement. For now, let us look at some examples.

(5.7) Examples. (i) $f(z) = \bar{z}$ is not $\mathbb{C}$-differentiable.

That is, $u(x,y) = x$ and $v(x,y) = -y$.

$$\frac{\partial u}{\partial x} = 1 \qquad \frac{\partial v}{\partial x} = 0$$

$$\frac{\partial u}{\partial y} = 0 \qquad \frac{\partial v}{\partial y} = -1$$

Cauchy-Riemann equations do not hold: $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ ✗

[They must <u>both</u> hold!] $\left(\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \text{ sure } \checkmark \right)$

(ii) $f(z) = |z|^2 = x^2 + y^2$.

In this case $u(x,y) = x^2 + y^2$ and $v(x,y) = 0$

$$\frac{\partial u}{\partial x} = 2x \qquad \frac{\partial v}{\partial x} = 0$$

$$\frac{\partial u}{\partial y} = 2y \qquad \frac{\partial v}{\partial y} = 0$$

C-R eq$^{\underline{n}}$s hold exactly at $(0,0)$ and nowhere else.

(iii) $f(z) = z^2 = (x^2 - y^2) + (2xy)i$

$u(x,y) = x^2 - y^2$; $v(x,y) = 2xy$

$$\left.\begin{aligned} \frac{\partial u}{\partial x} &= 2x = \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} &= -2y = -\frac{\partial v}{\partial x} \end{aligned}\right\}$$

C-R equations hold everywhere