(6.0) Recall: last time we introduced the notion of $C$-differentiable functions. A function $f(z)$ of a complex variable is said to be $C$-differentiable at $z_0 \in \mathbb{C}$ if

\[
(*) - \quad \lim_{h \to 0} \frac{f(z_0 + h) - f(z_0)}{h} \quad \text{exists, denoted by } f'(z_0) \quad \text{or } \quad \frac{df}{dz} \bigg|_{z = z_0}.
\]

Writing real and imaginary parts separately:

\[
f(z) = U(x, y) + V(x, y)i \quad ; \quad z_0 = x_0 + y_0i,
\]

\[(x = \text{Re}(z), \quad y = \text{Im}(z))\]

we proved that the existence of the limit $(*)$ above implies the following two statements:

- The partial derivatives $\frac{\partial U}{\partial x}, \frac{\partial U}{\partial y}, \frac{\partial V}{\partial x}, \frac{\partial V}{\partial y}$ exist at $(x_0, y_0)$.
  
  [Newton's notation: $u_x, u_y, v_x, v_y$]

- \[
U_x(x_0, y_0) = V_y(x_0, y_0)
\]

\[
U_y(x_0, y_0) = -V_x(x_0, y_0)
\]

Cauchy-Riemann equations

Moreover, in this case:

\[
f'(z_0) = U_x(x_0, y_0) + V_x(x_0, y_0)i
\]

\[
(= V_y(x_0, y_0) - U_y(x_0, y_0)i)
\]
(6.1) Today we will prove a converse of this result.

**Theorem.** Assume $u(x,y)$ and $v(x,y)$ are two (real-valued) functions of two real variables, such that the following hypotheses hold:

(i) $u$ and $v$ are continuous at $(x_0, y_0)$

(ii) $u_x, u_y, v_x, v_y$ exist and are continuous at $(x_0, y_0)$

(iii) Cauchy–Riemann equations hold \[ \begin{cases} u_x(x_0, y_0) = v_y(x_0, y_0) \\ u_y(x_0, y_0) = -v_x(x_0, y_0) \end{cases} \]

Then: \[ f(z) = u(x,y) + v(x,y)i \] is $C$-differentiable at \[ z_0 = x_0 + y_0i \]

and \[ f'(z_0) = u_x(x_0, y_0) + v_x(x_0, y_0)i \]

\[ = v_y(x_0, y_0) - u_y(x_0, y_0)i \]

A end of the statement

(6.2) Hypotheses (i) and (ii). The assumptions (i) and (ii) imposed on functions $u(x,y)$ and $v(x,y)$ are to make sure that we can use the following multivariable analogue of Taylor's theorem from Calculus III:
Let $g(x,y)$ be a real-valued function of two real variables, defined on an open disc near $(x_0,y_0)$.

Then

Let us assume that the assumptions (i) and (ii) of Theorem (6.1) hold for $g(x,y)$. Then:

$$g(x_0+a, y_0+b)$$

$$= g(x_0,y_0) + a \cdot g_x(x_0,y_0) + b \cdot g_y(x_0,y_0) + R(a,b)$$

where

$$\lim_{(a,b) \to (0,0)} \frac{R(a,b)}{\sqrt{a^2 + b^2}} = 0.$$  

[In plain English: the remainder term $R(a,b)$ goes to 0, as $\sqrt{a^2 + b^2} \to 0$, at least to the "second order".]

Remark. - This theorem can be easily proved from its one-variable counterpart (fix $(x_0,y_0), (a,b)$ and consider $p(t) = g(x_0+ta, y_0+t)$. We are not going to prove it here,
but only use it to finish the proof of Theorem (6.1).

(6.3) Proof of Theorem (6.1).

Let \( \alpha = U_x(x_0, y_0) = V_y(x_0, y_0) \) (by hypothesis (iii)).
\[
\beta = V_x(x_0, y_0) = -U_y(x_0, y_0)
\]

We want to prove that the following limit exists:
\[
\lim_{h \to 0} \frac{f(z_0 + h) - f(z_0)}{h}, \text{ and is equal to } \alpha + \beta i.
\]

(\( h \in \mathbb{C} \))

Let \( h = a + bi \). Then:
\[
f(z_0 + h) - f(z_0) = U(x_0 + a, y_0 + b) - U(x_0, y_0) + i \left( V(x_0 + a, y_0 + b) - V(x_0, y_0) \right)
\]

By Taylor's theorem (6.2), we know:
\[
U(x_0 + a, y_0 + b) - U(x_0, y_0) = a \alpha - b \beta + R_1(a, b), \quad V(x_0 + a, y_0 + b) - V(x_0, y_0) = a \beta + b \alpha + R_2(a, b), \text{ and both}
\]
\[
\frac{R_1(a, b)}{\sqrt{a^2 + b^2}}, \quad \frac{R_2(a, b)}{\sqrt{a^2 + b^2}} \to 0 \text{ as } (a, b) \to (0, 0).
\]

Thus,
\[
\frac{f(z_0 + h) - f(z_0)}{h} = (\alpha + \beta i)
\]
\[
= \frac{(a \alpha - b \beta) + i(a \beta + b \alpha) - (\alpha + \beta i)(a + bi) + R_1(a, b) + R_2(a, b)i}{h}
\]
\[
\frac{R_1(a,b) + R_2(a,b)i}{h}
\]

From this, we conclude:

\[
\left| \frac{f(z_0 + h) - f(z_0)}{h} - (\alpha + \beta i) \right| = \frac{|R_1(a,b) + R_2(a,b)i|}{|h|}
\]

\[
\leq \frac{|R_1(a,b)|}{|h|} + \frac{|R_2(a,b)|}{|h|} \rightarrow 0 \text{ as } h \rightarrow 0.
\]

(triangle ineq.)

(6.4) A more compact form of Cauchy-Riemann equations.

Notice: 

\[
\begin{align*}
\mathfrak{x} &= \text{Re}(z) = \frac{z + \overline{z}}{2} \\
\mathfrak{y} &= \text{Im}(z) = \frac{z - \overline{z}}{2i}
\end{align*}
\]

\[
\begin{align*}
z &= x + yi \\
\overline{z} &= x - yi
\end{align*}
\]

allow us to rewrite any expression involving \((x, y)\) as an expression involving \((\mathfrak{z}, \overline{\mathfrak{z}})\), and vice versa.

e.g. (i) \(x^2 + y^2 - (xy)\mathfrak{i} = \left(\frac{z + \overline{z}}{2}\right)^2 + \left(\frac{z - \overline{z}}{2i}\right)^2 - \left(\frac{z + \overline{z}}{2}\right)^2\left(\frac{z - \overline{z}}{2i}\right)\mathfrak{i}
\]

(ii) \(z^3 - 3z\overline{z} = (x + yi)^3 - 3(x - yi)\)
For a function \( f(x + yi) = u(x,y) + v(x,y)i \); C-Reg's

given as an expression in \( x, y \) variables

\[
\begin{align*}
\frac{\partial u}{\partial x} &= \frac{\partial v}{\partial y} \\
\frac{\partial u}{\partial y} &= -\frac{\partial v}{\partial x}
\end{align*}
\]

Rewritten in \( z, \bar{z} \) variables, \( \frac{\partial f}{\partial z} \) and \( \frac{\partial f}{\partial \bar{z}} \)
can be computed using the chain rule:

\[
\begin{align*}
\frac{\partial f}{\partial z} &= \frac{\partial u}{\partial z} + \frac{\partial v}{\partial z} i \\
&= \left( \frac{\partial u}{\partial x} \frac{\partial x}{\partial z} + \frac{\partial u}{\partial y} \frac{\partial y}{\partial z} \right) + \left( \frac{\partial v}{\partial x} \frac{\partial x}{\partial z} + \frac{\partial v}{\partial y} \frac{\partial y}{\partial z} \right)i \\
&= \left( u_x \cdot \frac{1}{2} + u_y \cdot \frac{1}{2i} \right) + \left( v_x \cdot \frac{1}{2} + v_y \cdot \frac{1}{2i} \right)i \\
&= \frac{1}{2} \left( u_x + v_y \right) + \frac{i}{2} \left( v_x - u_y \right)
\end{align*}
\]

\[
\begin{align*}
\frac{\partial f}{\partial \bar{z}} &= \frac{\partial u}{\partial \bar{z}} + \frac{\partial v}{\partial \bar{z}} i \\
&= \left( \frac{\partial u}{\partial x} \frac{\partial x}{\partial \bar{z}} + \frac{\partial u}{\partial y} \frac{\partial y}{\partial \bar{z}} \right) + \left( \frac{\partial v}{\partial x} \frac{\partial x}{\partial \bar{z}} + \frac{\partial v}{\partial y} \frac{\partial y}{\partial \bar{z}} \right)i \\
&= \left( u_x \cdot \frac{1}{2} + u_y \cdot \left( -\frac{1}{2i} \right) \right) + i \left( v_x \cdot \frac{1}{2} + v_y \cdot \left( -\frac{1}{2i} \right) \right) \\
&= \frac{1}{2} \left( u_x - v_y \right) + \frac{i}{2} \left( v_x + u_y \right)
\end{align*}
\]
Hence, we see that \( \frac{\partial f}{\partial z} = 0 \) is equivalent to

\[
U_x = V_y \quad \text{(real part of } \frac{\partial f}{\partial z})
\]

and \( U_y = -V_x \quad \text{(imaginary part of } \frac{\partial f}{\partial z}) \)

\[(6.5) \text{ Conclusion of (6.4).} \quad \] If a function of a complex variable is given to us in terms of \( z \) and \( \bar{z} \) (instead of the equivalent form in terms of \( x \) and \( y \)), then it is \( C \)-differentiable if and only if it does not depend on \( \bar{z} \).

E.g. (i) Determine whether \( f(z) = |z|^2 \) is \( C \)-differentiable.

Method I \((x, y)\)

\[
U = x^2 + y^2 \quad (= \text{Re}(f))
\]

\[
V = 0 \quad (= \text{Im}(f))
\]

\[
U_x = 2x \neq V_y = 0 \quad \text{(as functions)}
\]

Method II \((z, \bar{z})\)

\[
f(z) = z \cdot \bar{z}
\]

\[
\frac{\partial f}{\partial \bar{z}} = z \neq 0.
\]

Hence, \( \text{not } C \)-differentiable
(ii) \[ f(z) = \text{Im}(z) \]

\[ \text{Method I} \quad (x, y) \]

\[ u(x, y) = y \quad ; \quad v(x, y) = 0 \]

\[ u_x = 0 = v_y \]

\[ u_y = 1 \neq -v_x = 0 \]

\[ \frac{\partial f}{\partial \bar{z}} = \frac{z - \bar{z}}{2i} \neq 0 \]

\[ \text{Hence, NOT C-differentiable} \]

(iii) \[ f(z) = a_n z^n + a_{n-1} z^{n-1} + \ldots + a_1 z + a_0 \quad ; \quad \text{where} \]

\[ \text{polynomial function} \]

\[ a_0, a_1, \ldots, a_n \in \mathbb{C} \]

\[ \frac{\partial f}{\partial \bar{z}} = 0 \quad \text{Hence it is C-differentiable.} \]

\[ \text{[Remark: re-expressing it in (x, y) variables and verifying } u_x = v_y \text{ is EXTREMELY tedious!]} \]

Similarly, rational functions, such as \[ \frac{z^2 + 1 + i}{z - 3} \], are C-differentiable.