Lecture 6

(6.0) Recall: last time we introduced the notion of $\mathbb{C}$-differentiable functions. A function $f(z)$ of a complex variable is said to be <u>$\mathbb{C}$-differentiable at</u> $z_0 \in \mathbb{C}$ if

$$(*) - \boxed{\lim_{\substack{h \to 0 \\ (h \in \mathbb{C})}} \frac{f(z_0+h) - f(z_0)}{h}}$$

exists, denoted by $f'(z_0)$ or $\left.\frac{df}{dz}\right|_{z=z_0}$.

Writing real and imaginary parts separately:

$$f(z) = u(x,y) + v(x,y)i \quad ; \quad z_0 = x_0 + y_0 i,$$
$$(x = \mathrm{Re}(z),\ y = \mathrm{Im}(z))$$

we proved that the existence of the limit $(*)$ above implies the following two statements:

- The partial derivatives $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ exist at (x_0,y_0).

  [Newton's notation: u_x, u_y, v_x, v_y]

- $$\boxed{\begin{aligned} u_x(x_0,y_0) &= v_y(x_0,y_0) \\ u_y(x_0,y_0) &= -v_x(x_0,y_0) \end{aligned}}$$ Cauchy-Riemann equations

Moreover, in this case: $f'(z_0) = u_x(x_0,y_0) + v_x(x_0,y_0)i$
$(= v_y(x_0,y_0) - u_y(x_0,y_0)i)$

(6.1) Today we will prove a converse of this result.

<u>Theorem.</u> Assume $u(x,y)$ and $v(x,y)$ are two (real-valued) functions of two real variables, such that the following hypotheses hold:

(i) u and v are continuous at (x_0, y_0)

(ii) u_x, u_y, v_x, v_y exist and are continuous at (x_0, y_0)

(iii) Cauchy-Riemann equations hold $\begin{cases} u_x(x_0,y_0) = v_y(x_0,y_0) \\ u_y(x_0,y_0) = -v_x(x_0,y_0) \end{cases}$

Then: $f(z) = u(x,y) + v(x,y)i$ is <u>$\mathbb{C}$-differentiable at $z_0 = x_0 + y_0 i$</u>,
$[x = \mathrm{Re}(z),\ y = \mathrm{Im}(z)]$

and
$$f'(z_0) = u_x(x_0,y_0) + v_x(x_0,y_0)\, i$$
$$= v_y(x_0,y_0) - u_y(x_0,y_0)\, i$$

——— end of the statement ———

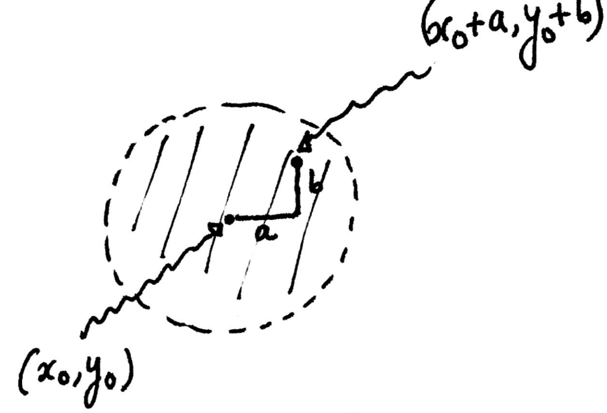
~~The~~ A proof is given in Section (6.3) below.

(6.2) <u>Hypotheses (i) and (ii)</u>. The assumptions (i) and (ii) imposed on functions $u(x,y)$ and $v(x,y)$ are to make sure that we can use the following multivariable analogue of Taylor's theorem from Calculus III:

Let $g(x,y)$ be a real-valued function of two real variables, defined on an open disc near (x_0, y_0).

~~then~~

Let us assume that the assumptions (i) and (ii) of Theorem (6.1) hold for $g(x,y)$. Then:



[$g(x,y)$'s domain contains this disc]

$$g(x_0+a,\, y_0+b) = g(x_0,y_0) + a\cdot g_x(x_0,y_0) + b\, g_y(x_0,y_0) + R(a,b)$$

where $\displaystyle\lim_{(a,b)\to(0,0)} \frac{R(a,b)}{\sqrt{a^2+b^2}} = 0$. [In plain english: the <u>remainder term $R(a,b)$</u> goes to 0, as $\sqrt{a^2+b^2} \to 0$, at least to the "second order".]

Remark. - This theorem can be easily proved from its one-variable counterpart (fix (x_0,y_0), (a,b) and consider $p(t) = g(x_0+ta,\, y_0+tb)$ ← function of one variable $t \in \mathbb{R}$.). We are not going to prove it here, but only use it to finish the proof of Theorem (6.1).

(6.3) <u>Proof of Theorem (6.1)</u>.

Let $\alpha = u_x(x_0, y_0) = v_y(x_0, y_0)$ (by hypothesis (iii)).

$\beta = v_x(x_0, y_0) = -u_y(x_0, y_0)$

We want to prove that the following limit exists:

$$\lim_{\substack{h \to 0 \\ (h \in \mathbb{C})}} \frac{f(z_0+h) - f(z_0)}{h}, \text{ and is equal to } \alpha + \beta i.$$

Let $h = a + bi$. Then: $f(z_0+h) - f(z_0) = u(x_0+a, y_0+b) - u(x_0, y_0)$
$+ i\left(v(x_0+a, y_0+b) - v(x_0, y_0)\right)$

By Taylor's theorem (6.2), we know:

$u(x_0+a, y_0+b) - u(x_0, y_0) = a\alpha - b\beta + R_1(a,b)$,

$v(x_0+a, y_0+b) - v(x_0, y_0) = a\beta + b\alpha + R_2(a,b)$, and both

$$\frac{R_1(a,b)}{\sqrt{a^2+b^2}}, \quad \frac{R_2(a,b)}{\sqrt{a^2+b^2}} \to 0 \quad \text{as} \quad (a,b) \to (0,0).$$

Thus,
$$\frac{f(z_0+h) - f(z_0)}{h} - (\alpha + \beta i)$$
$$= \frac{\cancel{(a\alpha - b\beta) + i(a\beta + b\alpha) - (\alpha + \beta i)(a+bi)} + R_1(a,b) + R_2(a,b)i}{h}$$

$$= \frac{R_1(a,b) + R_2(a,b)i}{h}$$

From this, we conclude:

$$\left| \frac{f(z_0+h) - f(z_0)}{h} - (\alpha+\beta i) \right| = \frac{|R_1(a,b) + R_2(a,b)i|}{|h|}$$

$$\underset{\text{(triangle ineq.)}}{\leq} \frac{|R_1(a,b)|}{|h|} + \frac{|R_2(a,b)|}{|h|} \to 0 \text{ as } h \to 0.$$

$\square$

(end of proof)

(6.4) A more compact form of Cauchy-Riemann equations.

Notice:

$$\boxed{\begin{aligned} x &= \operatorname{Re}(z) = \frac{z+\bar{z}}{2} \\ y &= \operatorname{Im}(z) = \frac{z-\bar{z}}{2i} \end{aligned}} \qquad \boxed{\begin{aligned} z &= x+yi \\ \bar{z} &= x-yi \end{aligned}}$$

allow us to rewrite any expression involving (x,y) as an expression involving $(z,\bar{z})$, and vice versa.

e.g. (i) $x^2 + y^2 - (x^2 y)i = \left(\frac{z+\bar{z}}{2}\right)^2 + \left(\frac{z-\bar{z}}{2i}\right)^2 - \left(\frac{z+\bar{z}}{2}\right)^2\left(\frac{z-\bar{z}}{2i}\right)i$

(ii) $z^3 - 3\bar{z} = (x+yi)^3 - 3(x-yi)$

For a function $f(x+yi) = u(x,y) + v(x,y)i$; C-R eq^{n}s
(given as an expression in x, y variables)

are $\boxed{\begin{aligned} \frac{\partial u}{\partial x} &= \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} &= -\frac{\partial v}{\partial x} \end{aligned}}$. Rewritten in $z, \bar{z}$ variables, $\frac{\partial f}{\partial z}$ and $\frac{\partial f}{\partial \bar{z}}$ can be computed using the chain rule :

- $$\begin{aligned} \frac{\partial f}{\partial z} &= \frac{\partial u}{\partial z} + \frac{\partial v}{\partial z} i \\ &= \left(\frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial z} + \frac{\partial u}{\partial y} \frac{\partial y}{\partial z}\right) + \left(\frac{\partial v}{\partial x} \frac{\partial x}{\partial z} + \frac{\partial v}{\partial y} \frac{\partial y}{\partial z}\right) i \\ &= \left(u_x \cdot \frac{1}{2} + u_y \cdot \frac{1}{2i}\right) + \left(v_x \cdot \frac{1}{2} + v_y \cdot \frac{1}{2i}\right) i \\ &= \frac{1}{2}\left(u_x + v_y\right) + \frac{i}{2}\left(v_x - u_y\right) \end{aligned}$$

- $$\begin{aligned} \frac{\partial f}{\partial \bar{z}} &= \frac{\partial u}{\partial \bar{z}} + \frac{\partial v}{\partial \bar{z}} i \\ &= \left(\frac{\partial u}{\partial x} \frac{\partial x}{\partial \bar{z}} + \frac{\partial u}{\partial y} \frac{\partial y}{\partial \bar{z}}\right) + \left(\frac{\partial v}{\partial x} \frac{\partial x}{\partial \bar{z}} + \frac{\partial v}{\partial y} \frac{\partial y}{\partial \bar{z}}\right) i \\ &= \left(u_x \cdot \frac{1}{2} + u_y \cdot \left(\frac{-1}{2i}\right)\right) + i\left(v_x \cdot \frac{1}{2} + v_y \cdot \left(\frac{-1}{2i}\right)\right) \\ &= \frac{1}{2}\left(u_x - v_y\right) + \frac{i}{2}\left(v_x + u_y\right) \end{aligned}$$

Hence, we see that $\boxed{\dfrac{\partial f}{\partial \bar{z}} = 0}$ is equivalent to

$$\boxed{\begin{array}{l} u_x = v_y \quad \left(\text{real part of } \dfrac{\partial f}{\partial \bar{z}}\right) \\ \text{and } \; u_y = -v_x \quad \left(\text{imaginary part of } \dfrac{\partial f}{\partial \bar{z}}\right) \end{array}}$$

(6.5) <u>Conclusion of (6.4).</u> – If a function of a complex variable is given to us in terms of z and $\bar{z}$ (instead of the equivalent form in terms of x and y), then <u>it is</u> <u>$\mathbb{C}$-differentiable</u> if and only if <u>it does not depend on $\bar{z}$</u>.

e.g. (i) Determine whether $f(z) = |z|^2$ is $\mathbb{C}$-differentiable

Method I (x, y)

$u = x^2 + y^2 \quad (= \mathrm{Re}(f))$

$v = 0 \quad (= \mathrm{Im}(f))$

$u_x = 2x \neq v_y = 0$

(as functions)

Method II $(z, \bar{z})$

$f(z) = z \cdot \bar{z}$

$\dfrac{\partial f}{\partial \bar{z}} = z \neq 0.$

Hence, <u>NOT</u> $\mathbb{C}$-differentiable

(ii) $f(z) = \operatorname{Im}(z)$

Method I (x, y)

$u(x,y) = y$; $v(x,y) = 0$

$u_x = 0 = v_y$

$u_y = 1 \neq -v_x = 0$

Method II $(z, \bar{z})$

$$f(z) = \frac{z - \bar{z}}{2i}$$

$$\frac{\partial f}{\partial \bar{z}} = \frac{-1}{2i} \neq 0$$

Hence, NOT $\mathbb{C}$-differentiable

(iii) $f(z) = a_n z^n + a_{n-1} z^{n-1} + \dots + a_1 z + a_0$; where $a_0, a_1, \dots, a_n \in \mathbb{C}$ are fixed complex numbers.

[polynomial function]

$\frac{\partial f}{\partial \bar{z}} = 0$. Hence it is $\mathbb{C}$-differentiable.

[Remark. - re-expressing it in (x,y) variables and verifying $u_x = v_y$, $u_y = -v_x$ is EXTREMELY tedius!]

Similarly, rational functions, such as $\frac{z^2 + 1 + i}{z - 3}$, are $\mathbb{C}$-differentiable.