Lecture 7

(7.0) Recall: last time we proved the sufficiency of the Cauchy-Riemann equations.

- $f(z) = u(x,y) + v(x,y)i$ is $\mathbb{C}$-differentiable at $z_0 = x_0 + y_0 i$
  $[x = \mathrm{Re}(z),\ y = \mathrm{Im}(z)]$

  $\Rightarrow$ $\boxed{\begin{aligned} u_x &= v_y \\ u_y &= -v_x \end{aligned}}$ at (x_0, y_0) $\boxed{\begin{aligned} f'(z_0) &= u_x(x_0,y_0) + v_x(x_0,y_0)i \\ &= v_y(x_0,y_0) - u_y(x_0,y_0)i \end{aligned}}$

- $u(x,y), v(x,y)$ continuous, with continuous first partial derivatives at (x_0, y_0) such that $\boxed{\begin{aligned} u_x &= v_y \\ u_y &= -v_x \end{aligned}}$ at (x_0,y_0), implies that

  $f(z) = u(x,y) + v(x,y)i$ is $\mathbb{C}$-differentiable at $z_0 = x_0 + y_0 i$.

- Alternate form of Cauchy-Riemann equations, in $(z, \bar{z})$ coordinates: $\boxed{\dfrac{\partial f}{\partial \bar{z}} = 0}$

(7.1) If $f(z)$ is a complex-valued function, defined on an open set $\Omega \subseteq \mathbb{C}$, then we say $f(z)$ is <u>$\mathbb{C}$-differentiable</u> if it is $\mathbb{C}$-differentiable at all $z_0 \in \Omega$.

Remark. - The alternate form (namely, $\frac{\partial f}{\partial \bar{z}} = 0$) of Cauchy-Riemann equations can be viewed as saying that $f(z)$ is genuinely a function of <u>one</u> variable - since it is not supposed to depend on the "other one" - i.e., $\bar{z}$.

Thus we are justified in writing $\frac{df}{dz}$, instead of more pedantic $\frac{\partial f}{\partial z}$ for <u>the derivative of f with respect to z</u>.

(7.2) <u>Laplace equation and harmonic functions</u>.

Assume that $f(z) = u(x,y) + v(x,y)i$ is a $\mathbb{C}$-differentiable function. We know that the Cauchy-Riemann equations must hold :

$$u_x = v_y \qquad u_y = -v_x$$

. Let us assume that u and v are <u>twice differentiable</u>

[i.e. 2^{nd} order partial derivatives exist, and under a mild assumption $\frac{\partial}{\partial x} \cdot \frac{\partial}{\partial y} = \frac{\partial}{\partial y} \frac{\partial}{\partial x}$ - Clairaut's Thm. from Calculus]

Then:

$$\begin{aligned} u_{xx} &= v_{yx} \\ u_{yy} &= -v_{xy} \\ \hline u_{xx} + u_{yy} &= 0 \end{aligned}$$

[$\frac{\partial (u_x)}{\partial x}$ or $\frac{\partial^2 u}{\partial x^2}$]

Similarly, we get $v_{xx} + v_{yy} = 0$.

<u>Definition</u>. – For a real-valued function of two real variables, say $u(x,y)$, <u>the Laplace equation</u> is :

$$\boxed{\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0}$$

A function satisfying the Laplace equation is called a <u>harmonic function</u>.

Thus we see that :

$u(x,y) + v(x,y)i$ is $\mathbb{C}$-differentiable	$\Longrightarrow$	u and v are harmonic functions

(& u & v are twice differentiable)

(7.3) Examples. – (1) Show that there is no $\mathbb{C}$-differentiable $f(z)$ such that $\mathrm{Re}(f(z)) = x^3$.

$$u(x,y) = x^3 \Rightarrow u_x = 3x^2 \; ; \; u_{xx} = 6x$$
$$u_y = 0 \; ; \; u_{yy} = 0$$

$u_{xx} + u_{yy} = 6x \neq 0$. Thus $u(x,y)$ is <u>NOT</u> harmonic

Hence there is no $v(x,y)$ such that $u(x,y) + v(x,y)i$ is $\mathbb{C}$-differentiable.

(2) $u(x,y) = 2x+y$ is a solution of the Laplace equation.

Let us find $v(x,y)$ that makes $u(x,y)+v(x,y)i$ into a $\mathbb{C}$-differentiable function.

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} = -1 \Rightarrow v(x,y) = -x + g(y)$$

$$\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} = 2 \Rightarrow \frac{\partial}{\partial y}\left(-x+g(y)\right) = 2 \Rightarrow g'(y)=2$$

$$\Rightarrow g(y) = 2y + C \quad (C \in \mathbb{R} \text{ some constant}).$$

Thus, $f(z) = (2x+y) + (-x+2y+c)i = 2z - iz + C$

[Sanity check: it does not depend on $\bar{z}$ ✓]

(3) $u(x,y) = e^{x}\cos(y)$ is a solution of the Laplace equation

Again let us find $v(x,y)$ such that Cauchy-Riemann equations hold:

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} = e^{x}\sin(y) \Rightarrow v(x,y) = e^{x}\sin(y) + g(y)$$

$$\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} = e^{x}\cos(y)$$

i.e. $e^{x}\cos(y) + g'(y) = e^{x}\cos(y) \Rightarrow g'(y) = 0 \Rightarrow g(y) = C$

Thus, $v(x,y) = e^{x}\sin(y) + C$.

(7.4) Properties of $\frac{d}{dz}$, directly from definition

$$\boxed{f'(z_0) = \lim_{h\to 0} \frac{f(z_0+h) - f(z_0)}{h}}$$

(1) $\left(f(z)+g(z)\right)' = f'(z) + g'(z)$

(2) $\left(c\cdot f(z)\right)' = c\cdot f'(z)$

VERY easy to prove.

(3) $\frac{d}{dz}(z^n) = n z^{n-1}$. $(n \in \mathbb{Z}_{\geq 0})$.

Proof $\quad \frac{d}{dz}(z^n) = \lim_{h\to 0} \frac{(z+h)^n - z^n}{h}$

$$= \lim_{h\to 0} \frac{1}{h}\left(\cancel{z^n} + n z^{n-1} h + \binom{n}{2} z^{n-2} h^2 + \dots + h^n \cancel{- z^n}\right)$$

$$= \lim_{h\to 0} \left(n z^{n-1} + h(\dots)\right) = n z^{n-1} \quad \square$$

(4) $\frac{d}{dz}(z^{-1}) = \lim_{h\to 0} \frac{1}{h}\left(\frac{1}{z+h} - \frac{1}{z}\right) = \lim_{h\to 0} \frac{1}{h}\left(\frac{z-z-h}{(z+h)z}\right)$

$$= \lim_{h\to 0}\left(\frac{-1}{(z+h)z}\right) = \frac{-1}{z^2}.$$

(7.5) Assume that $f(z)$ is defined on an open set Ω, and is $\mathbb{C}$-differentiable. Let $z_0 \in \Omega$ and $h \in \mathbb{C}$ be small enough so that $z_0 + h$ is still in Ω.

[More precisely, since Ω is open, there is some $r \in \mathbb{R}_{>0}$ such that $D(z_0; r) \subset \Omega$. Assume that $|h| < r$.]

Write: $f(z_0 + h) - f(z_0) - h f'(z_0) = R(h)$.

Then, by definition of $f'(z_0)$, we have:

$$\boxed{\lim_{h \to 0} \frac{R(h)}{h} = 0}$$

(analogue of "linear approximation").

(7.6) Product and chain rule.

(i) $(f(z)\, g(z))' = f'(z)\, g(z) + f(z)\, g'(z)$.

Proof.
$$f(z+h)\, g(z+h) = \left(f(z) + h f'(z) + R_1(h)\right) \cdot \left(g(z) + h g'(z) + R_2(h)\right)$$
$$= f(z)\, g(z) + h\left(f'(z)\, g(z) + f(z)\, g'(z)\right) + R(h),$$

(where $R(h) = f(z) R_2(h) + g(z) R_1(h) + h^2 f'(z) g'(z) + h\left(f'(z) R_2(h) + g'(z) R_1(h)\right) + R_1(h) R_2(h)$ satisfies

$$\lim_{h \to 0} \frac{R(h)}{h} = 0 \,.\ )$$

Thus, $$\lim_{h \to 0} \frac{f(z+h)\,g(z+h) - f(z)g(z)}{h} = f'(z)g(z) + f(z)g'(z).$$

(2) Chain rule. $$\frac{d}{dz}\Big(f(g(z))\Big) = f'(g(z))\,g'(z)$$

[I am omitting the proof of this.]

(7.7) <u>L'Hospital rule</u> If $f(z_0) = g(z_0) = 0$ and $f'(z)$, $g'(z)$ exist and $g'(z_0) \neq 0$; then

$$\lim_{z \to z_0} \frac{f(z)}{g(z)} = \frac{f'(z_0)}{g'(z_0)}$$

Proof. $$\lim_{z \to z_0} \frac{f(z)}{g(z)} = \lim_{h \to 0} \frac{f(z_0+h)}{g(z_0+h)}$$

$$= \lim_{h \to 0} \left(\frac{\cancelto{0}{f(z_0)} + h f'(z_0) + R_1(h)}{\cancelto{0}{g(z_0)} + h g'(z_0) + R_2(h)} \right)$$

(see Section 7.5)

$$= \lim_{h \to 0} \left(\frac{f'(z_0) + \frac{R_1(h)}{h}}{g'(z_0) + \frac{R_2(h)}{h}} \right) = \frac{f'(z_0)}{g'(z_0)} \,. \quad \square$$