Lecture 8

Exponential function and trigonometric functions

(8.0) Recall - from Calculus I - functions of one real-variable

(i) $e \in \mathbb{R}_{>0}$ is defined as :

[Euler's constant ; ~~L~~ Leonhard Euler 1707-1783, Basel, Switzerland.]

$$\boxed{e = \lim_{\substack{n \to \infty \\ (n \in \mathbb{Z}_{\geq 1})}} \left(1 + \frac{1}{n}\right)^n}$$

[This limit was studied earlier by Jacob Bernoulli 1655-1705.]

$= 2.71828 \ldots$ (you can get accuracy up to 4 decimal places by setting $n = 100000$.)

Euler found an alternate way to compute this limit

$$\boxed{e = \lim_{n \to \infty} \left(1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \ldots + \frac{1}{n!}\right)}$$

(accuracy up to 4 decimal places : even at $n = 7$)

(ii) The function $\underset{(x \in \mathbb{R})}{e^x}$ can then be defined in these two analoguous ways :

$$e^x = \lim_{n \to \infty} \left(1 + \frac{x}{n}\right)^n = \lim_{n \to \infty} \left(1 + \frac{x}{1!} + \frac{x^2}{2!} + \ldots + \frac{x^n}{n!}\right)$$

(iii) Fundamental properties of e^x:

$$e^0 = 1; \qquad e^{x_1} \cdot e^{x_2} = e^{x_1+x_2}; \quad \frac{d}{dx} e^x = e^x; \ (e^x)^a = e^{ax}.$$

Remark. - We can in fact define e^x as the only function $f(x)$ such that $\left\{ \begin{array}{l} f'(x) = f(x) \\ f(0) = 1 \end{array} \right\}$ (Euler's point of view).

(8.1) The function e^z $(z \in \mathbb{C})$.

Let us define $\boxed{e^z = e^x(\cos(y) + \sin(y) i)}$. The domain of this function is $\Omega = \mathbb{C}$ (entire complex plane). Moreover, as you checked in HW2 [Problem 12 (2)], Cauchy-Riemann equations hold:

$$u(x,y) = e^x \cos(y) \Rightarrow u_x = e^x \cos(y), \quad u_y = -e^x \sin(y)$$

$$v(x,y) = e^x \sin(y) \Rightarrow v_x = e^x \sin(y), \quad v_y = e^x \cos(y)$$

Thus, $u_x = v_y$ and $u_y = -v_x$ for every (x,y),

and: $$\frac{d}{dz} e^z = u_x + i v_x = e^x (\cos(y) + \sin(y) i)$$

[Problem Set 3; # 2(d)]

$$= e^z.$$

Hence e^z is $\mathbb{C}$-differentiable everywhere, $\boxed{\frac{d}{dz} e^z = e^z}$.

It is also easy to see that $e^0 = 1$. In fact, for $z = x \in \mathbb{R}$

our definition recovers the real-valued version e^x.

(8.2) $$\boxed{e^{z_1} \cdot e^{z_2} = e^{z_1 + z_2}} \qquad (z_1 = x_1 + y_1 i \,;\; z_2 = x_2 + y_2 i)$$

Proof.
$$e^{z_1} \cdot e^{z_2} = e^{x_1}\left(\cos(y_1) + \sin(y_1) i\right) e^{x_2}\left(\cos(y_2) + \sin(y_2) i\right)$$

$$= e^{x_1 + x_2}\Big[\left(\cos(y_1)\cos(y_2) - \sin(y_1)\sin(y_2)\right) + \left(\cos(y_1)\sin(y_2) + \sin(y_1)\cos(y_2)\right) i\Big]$$

By the analoguous property of e^x ($x \in \mathbb{R}$)

$$= e^{x_1 + x_2}\left(\cos(y_1 + y_2) + \sin(y_1 + y_2) i\right)$$

By addition formulae for cosine & sine

$$= e^{z_1 + z_2}. \qquad \square$$

(8.3) For $z = \overset{\theta}{\cancel{y}} i$, our definition is the famous

Euler's formula:

$$\boxed{e^{i\theta} = \cos(\theta) + \sin(\theta) i \qquad (\theta \in \mathbb{R})}$$

Thus, from now onwards, we will write polar form of

$z \in \mathbb{C}\,;\; z \neq 0$ as

$$\boxed{z = r e^{i\theta}} \qquad (r = |z|,\; \theta = \arg(z))$$

Recommended reading*: Feynman, Lectures on Physics, Chapter 22 - Algebra (see eq^n (22.9)).

*optional

(8.4) <u>Periodicity of e^z.</u>

$$\boxed{e^{z_1} = e^{z_2} \text{ if and only if } z_1 - z_2 \in 2\pi i\mathbb{Z}}$$

- For $z = x+yi$, $|e^z| = e^x \in \mathbb{R}_{>0}$. Thus $\underline{e^z \neq 0}$.

  Moreover $\frac{1}{e^z} = e^{-z}$.

- $e^z = 1 \iff e^x = 1$, i.e. $x = 0$.

  and

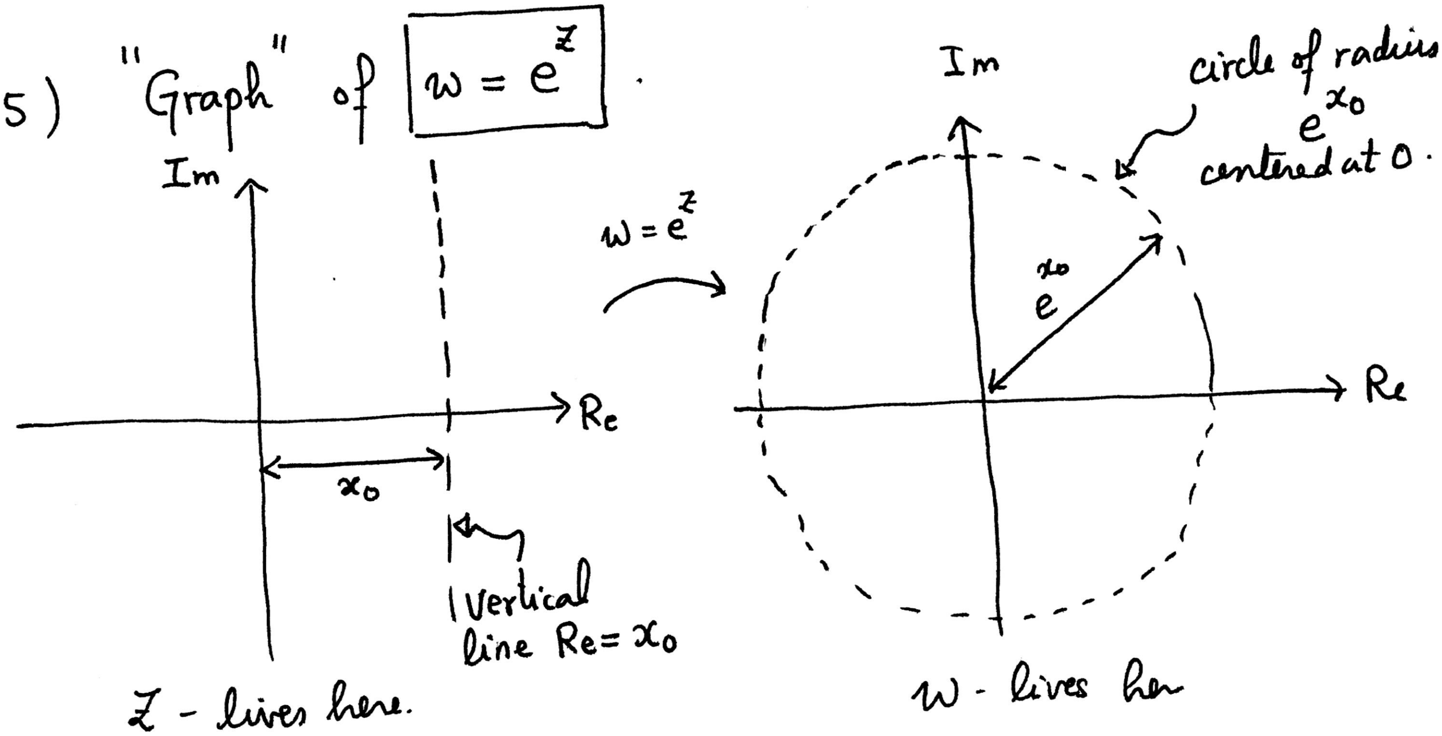
  $\boxed{\begin{array}{l}\cos(y) = 1 \\ \sin(y) = 0\end{array}}$ i.e. $y \in 2\pi\mathbb{Z}$.

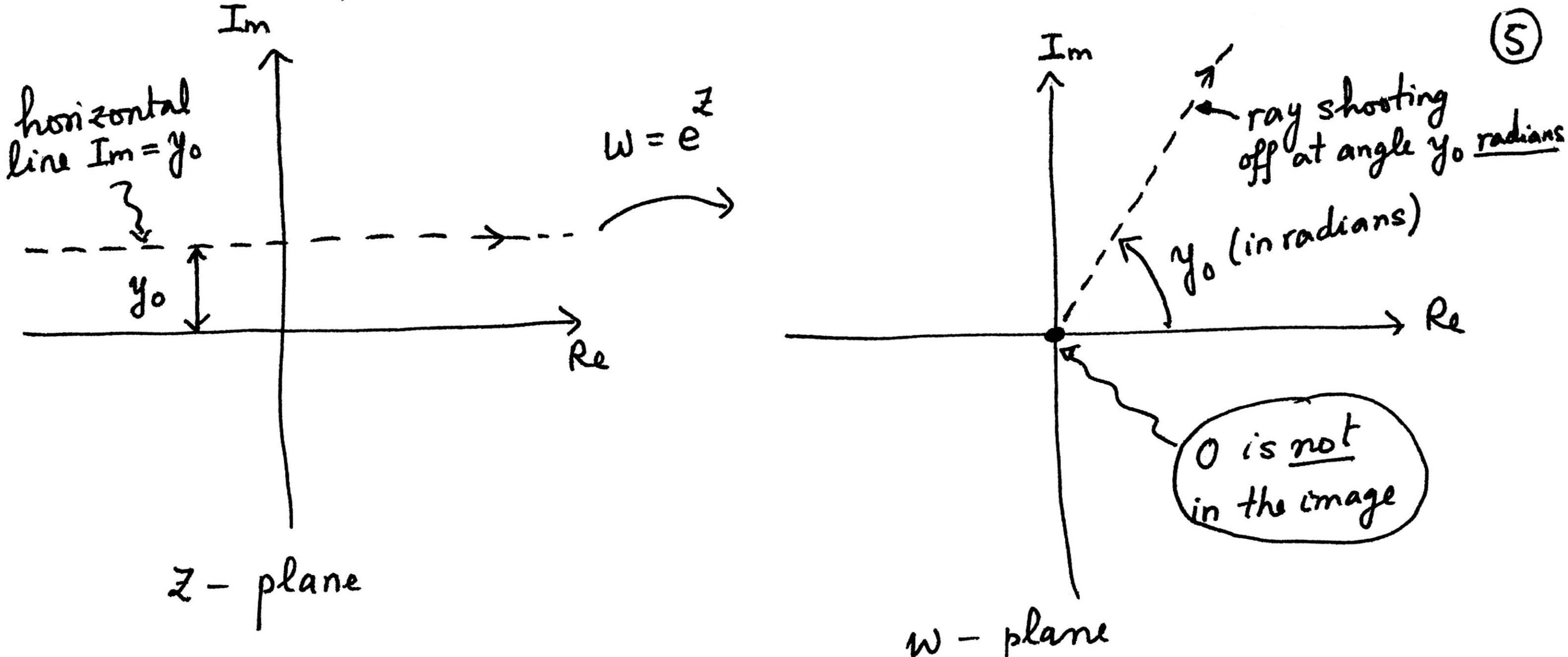
Thus, $e^{z_1} = e^{z_2}$ if and only if $e^{z_1 - z_2} = 1$

$\iff z_1 - z_2 = (2\pi n)i$ for some $n \in \mathbb{Z}$

(i.e., $z_1 - z_2 \in 2\pi i\mathbb{Z}$) $\square$

(8.5) "Graph" of $\boxed{w = e^z}$.





- $w = e^z$ transforms $\left\{ \begin{array}{l} \text{vertical lines to circles} \\ \text{horizontal lines to rays} \end{array} \right\}$.

(8.6) sin(z) and cos(z).

Note: for $\theta \in \mathbb{R}$, Euler's formula gives:

$$\begin{aligned} e^{i\theta} &= \cos(\theta) + \sin(\theta) i \\ e^{-i\theta} &= \cos(\theta) - \sin(\theta) i \end{aligned} \implies \begin{aligned} \cos(\theta) &= \frac{e^{i\theta} + e^{-i\theta}}{2} \\ \sin(\theta) &= \frac{e^{i\theta} - e^{-i\theta}}{2i} \end{aligned}$$

This allows us to define, for any $z \in \mathbb{C}$

$$\boxed{\cos(z) = \frac{e^{iz} + e^{-iz}}{2}} \qquad \boxed{\sin(z) = \frac{e^{iz} - e^{-iz}}{2i}}$$

(8.7) $\quad \dfrac{d}{dz}\sin(z) = \cos(z)$ and $\dfrac{d}{dz}\cos(z) = -\sin(z)$.

Proof. - $\quad \dfrac{d}{dz}\sin(z) = \dfrac{d}{dz}\left(\dfrac{e^{iz} - e^{-iz}}{2i}\right)$

$$= \frac{1}{2i}\left(e^{iz}\cdot i - e^{-iz}(-i)\right) \quad \left(\text{by Chain rule } \& \ \frac{de^{z}}{dz} = e^{z}\right)$$

$$= \frac{e^{iz} + e^{-iz}}{2} = \cos(z). \qquad \square$$

(The other one is similar, hence omitted.)

(8.8) $$\boxed{\begin{aligned} \sin(z_1+z_2) &= \sin(z_1)\cos(z_2) + \cos(z_1)\sin(z_2) \\ \cos(z_1+z_2) &= \cos(z_1)\cos(z_2) - \sin(z_1)\sin(z_2) \end{aligned}}$$

Proof (only of the first). -

$$\sin(z_1)\cos(z_2) + \cos(z_1)\sin(z_2) = \frac{\left[\left(e^{iz_1} - e^{-iz_1}\right)\left(e^{iz_2} + e^{-iz_2}\right) + \left(e^{iz_1} + e^{-iz_1}\right)\left(e^{iz_2} - e^{-iz_2}\right)\right]}{4i}$$

$$= \frac{1}{4i}\left(e^{i(z_1+z_2)} \cancel{+ e^{i(z_1-z_2)}} \cancel{- e^{-i(z_1-z_2)}} - e^{-i(z_1+z_2)} + e^{i(z_1+z_2)} \cancel{- e^{i(z_1-z_2)}} \cancel{+ e^{-i(z_1-z_2)}} - e^{-i(z_1+z_2)}\right)$$

$$= \frac{e^{i(z_1+z_2)} - e^{-i(z_1+z_2)}}{2i} = \sin(z_1+z_2).$$

□

(8.9) $\sin(z+2\pi k) = \sin(z)$
$\cos(z+2\pi k) = \cos(z)$ for every $k \in \mathbb{Z}$.

[This is obvious from $e^{i(2\pi k)} = 1$.]

$$\boxed{\sin^2(z) + \cos^2(z) = 1}$$

Proof.- $(A+A^{-1})^2 = A^2 + A^{-2} + 2$.
$(A-A^{-1})^2 = A^2 + A^{-2} - 2$.

Using these, we get:

$$\sin^2(z) + \cos^2(z) = \frac{e^{2iz} + e^{-2iz} - 2}{4(-1)} + \frac{e^{2iz} + e^{-2iz} + 2}{4} = \frac{4}{4} = 1.$$

□

(8.10) Point of departure.- Recall, for $\theta \in \mathbb{R}$, $|\sin(\theta)| \leq 1$.
$|\cos(\theta)| \leq 1$.

However for $z \in \mathbb{C}$, $|\sin(z)|$ and $|\cos(z)|$ are not bounded by any constant. It is easy to see, since for $z = iy$

$$\cos(iy) = \frac{e^{-y} + e^{y}}{2} \to \infty \text{ as } y \to +\infty.$$