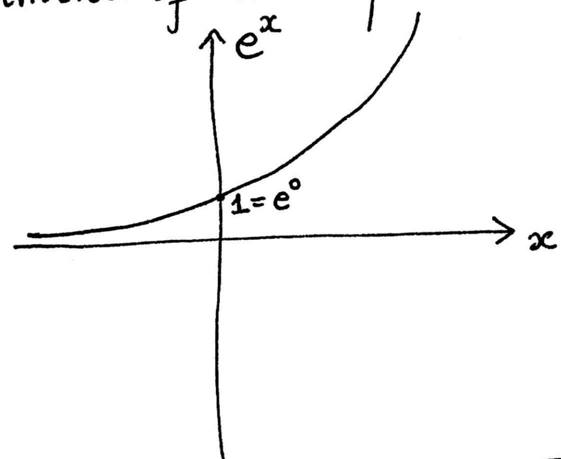
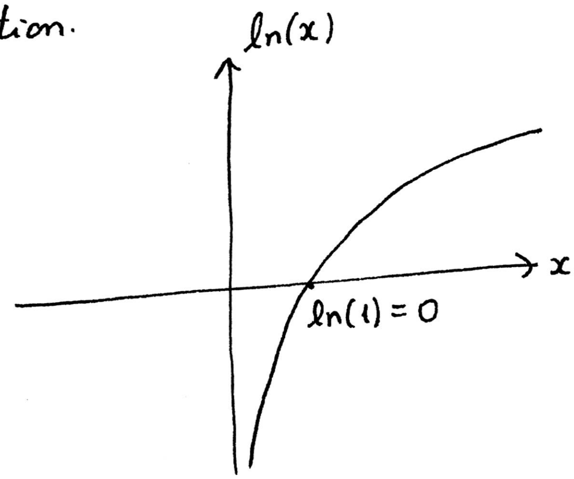
Lecture 9

Logarithm and general exponents

(9.0) Recall - for function of real variables:

- $\underline{\ln(x)}$ (natural logarithm of x) is defined as the inverse of the exponential function.





$$\boxed{b = \ln(a) \iff a = e^b \quad (a \in \mathbb{R}_{>0})}$$

- for $x \in \mathbb{R}_{>0}$ and $a \in \mathbb{R}$ (any); x^a is then defined as:

$$\boxed{x^a = e^{a \ln(x)}}$$

<u>Properties of $\ln(x)$</u>:

"inverse" of the properties of e^x - see (8.0) page 2

(i) $\ln(x_1 x_2) = \ln(x_1) + \ln(x_2)$

(ii) $\ln(1) = 0$

(iii) $\dfrac{d}{dx} \ln(x) = \dfrac{1}{x}$

(iv) $\ln(x^a) = a \ln(x)$

(9.1) The difficulty in defining $\ln(z)$ for $z \in \mathbb{C}$ lies in the fact that $e^{z} = e^{z+2\pi i} = e^{z+4\pi i} = \cdots$

[recall: this goes back to the ambiguity in defining $\arg(z)$ - it is only up to $\pm 2\pi$.]

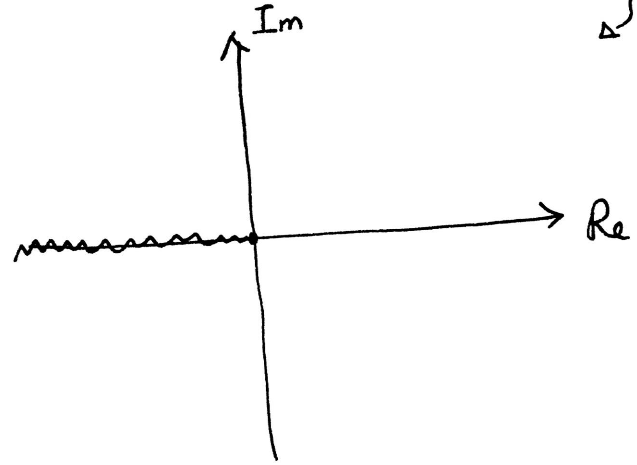
In order to resolve this ambiguity - we need to make a choice.

[Ambiguity: $\alpha, \beta \in \mathbb{C}$; $\beta = e^{\alpha} = e^{\alpha + 2\pi k i}$ should mean $\ln(\beta) = \alpha + 2\pi k i$? Which k should we take?]

Definition. $\boxed{\ln(z) = \ln(|z|) + \arg(z)\, i}$ defined on

- $\Omega = \mathbb{C} \setminus \mathbb{R}_{\leq 0} = \{ z \in \mathbb{C} \mid z \text{ is not a negative real or zero} \}$
- $\arg(z) \in (-\pi, \pi)$

  [$\pm\pi$ NOT included]

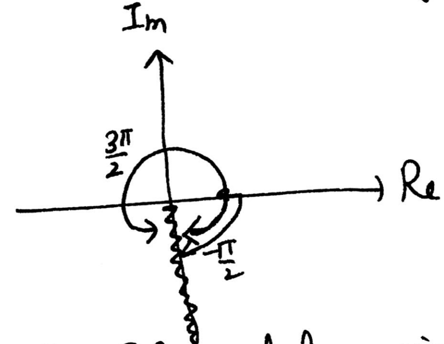


[see HW 3 problem 7.]

(9.2) Other choices can also be made, for instance, you may want to define your own logarithm

$$\underline{\mathrm{Log}}(z) = \ln(|z|) + \underline{\mathrm{Arg}}(z)i \quad \text{where } -\frac{\pi}{2} < \underline{\mathrm{Arg}}(z) < \frac{3\pi}{2}$$

for $z \in \mathbb{C} \setminus \mathbb{R}_{\leq 0}\, i$.



But we will never be able to define a logarithm on the whole $\mathbb{C} \setminus \{0\}$; while maintaining continuity. [e^z is NEVER 0.] For example, for the function $\underline{\mathrm{Log}}$, if it were possible to define it also on $\mathbb{R}_{<0}\, i$, we would have

$$\underline{\mathrm{Log}}(-i) = -\frac{\pi}{2} \quad \text{(approached from left)}$$
$$= \frac{3\pi}{2} \quad \text{(approached from right)}.$$

(9.3) $\boxed{\ln(z) = \ln(|z|) + \arg(z)\, i}$ $\quad -\pi < \arg(z) < \pi$.

Domain: $\mathbb{C} - \mathbb{R}_{\leq 0}$.

Cauchy-Riemann equations.

$$u(x,y) = \ln\left(\sqrt{x^2+y^2}\right) = \frac{1}{2}\ln(x^2+y^2)$$

$$\Rightarrow u_x = \frac{x}{x^2+y^2} \quad \text{and} \quad u_y = \frac{y}{x^2+y^2}.$$

$v(x,y) = \arctan\left(\frac{y}{x}\right)$ [For simplicity, assume $x = \mathrm{Re}(z) > 0$ here.]

$$\Rightarrow v_x = \frac{1}{1+\frac{y^2}{x^2}} \cdot \frac{-y}{x^2} = \frac{-y}{x^2+y^2} \quad \left(\text{recall } \frac{d}{dt}\arctan(t) = \frac{1}{1+t^2}\right)$$

$$v_y = \frac{1}{1+\frac{y^2}{x^2}} \cdot \frac{1}{x} = \frac{x}{x^2+y^2}.$$

Thus, Cauchy-Riemann equations hold and hence $\ln(z)$ is $\mathbb{C}$-differentiable. Moreover,

$$\frac{d}{dz}\ln(z) = u_x + i v_x = \frac{x-yi}{x^2+y^2} = \frac{\bar{z}}{z\cdot\bar{z}} = \frac{1}{z}.$$

$$\boxed{\frac{d}{dz}\ln(z) = \frac{1}{z}}.$$

It is clear that for $z = x \in \mathbb{R}_{>0}$, $\ln(z) =$ original $\ln(x)$ (real fn.)

In particular, we still have $\ln(1) = 0$.

~~(9.4) $\ln(z_1 z_2) = \ln(z_1) + \ln(z_2)$ up to $2\pi\mathbb{Z}i$~~

(9.4) In order to consider composition of $\ln(z)$ with other functions, we will have to be careful, since we do not have $\ln(z \in \mathbb{R}_{\leq 0})$.

e.g. • $\ln(z+2)$ is defined on $\Omega = \{ z \in \mathbb{C} \mid z+2 \notin \mathbb{R}_{\leq 0} \}$
$= \mathbb{C} \setminus \mathbb{R}_{\leq -2}$.

• $\ln(\alpha \cdot z)$ $(\alpha \in \mathbb{C}; \alpha \neq 0)$ is defined on $\{ z \in \mathbb{C} \mid \alpha z \notin \mathbb{R}_{\leq 0} \}$

i.e. $\alpha z \neq -t$ $(t \in \mathbb{R}_{>0})$

$z \neq \dfrac{-t\bar{\alpha}}{|\alpha|^2}$

$= \mathbb{C} \setminus \bar{\alpha} \cdot \mathbb{R}_{\leq 0}$.

(9.5)* Remarks.— Often, choosing "intuitive clarity" over mathematical precision, people write:

$$\text{"}\log\text{"}(z) = \ln(|z|) + (\arg(z) + 2\pi k)i \quad (k \in \mathbb{Z})$$

is a "multi-valued function". The notion is clearly absurd (a function takes one input and gives one output, so, how can a function be multi-valued?). But it does allow us to write "log"$(z_1 z_2)$ = "log"(z_1) + "log"(z_2) without any long explanations. In order to turn this absurdity into mathematical reality, Riemann introduced the idea, now known as Riemann Surfaces (beyond our scope - for now!)

*optional

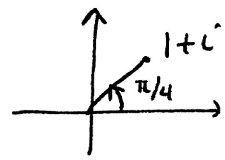
(9.6) $$\boxed{e^{\ln(z)} = z}$$

If $z = r \cdot e^{i\theta}$ then $\ln(z) = \ln(r) + i\theta$

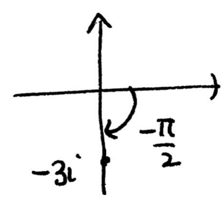
$(-\pi < \theta < \pi)$ $\Rightarrow e^{\ln(z)} = e^{\ln(r)} \cdot e^{i\theta} = r \cdot e^{i\theta} = z.$ $\square$

(9.7) Examples.

(1) $\ln(1+i) = \ln(\sqrt{2}) + \frac{\pi}{4} i$



(2) $\ln(-3i) = \ln(3) - \frac{\pi}{2} i$



(3) $\frac{d}{dz} \ln(z^2+3) = \frac{2z}{z^2+3}$ wherever it makes sense!

(9.8) $\boxed{z^\alpha \text{ where } \alpha \in \mathbb{C}}$ is defined as

$$\boxed{z^\alpha = e^{\alpha \ln(z)}}$$ again for $z \in \mathbb{C} - \mathbb{R}_{\leq 0}$.

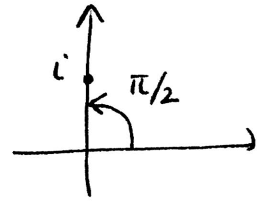
Ex. $\frac{d}{dz}(z^\alpha) = \frac{d}{dz}\left(e^{\alpha \ln(z)}\right) = e^{\alpha \ln(z)} \cdot \frac{\alpha}{z}$

$= z^\alpha \cdot \frac{\alpha}{z} = \alpha \cdot z^{\alpha - 1}.$

Example. $\boxed{i^i = e^{-\frac{\pi}{2}}}$.

$$i^i = e^{i\ln(i)} = e^{i(0+\frac{\pi}{2}i)}$$

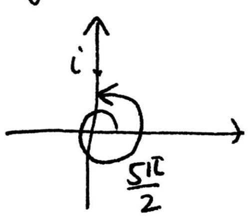
$$= e^{-\frac{\pi}{2}}.$$



$$\ln(i) = \ln(1) + \tfrac{\pi}{2}i = \tfrac{\pi}{2}i$$

(This calculation clearly depends on our chosen definition of $\ln(z)$.

For, if you had defined $\underline{\underline{Log}}(z)$ that makes $\underline{\underline{Log}}(i) = \frac{5\pi}{2}i$, the answer would turn out to be $e^{-\frac{5\pi}{2}}$.



The only thing that can be said, independent of choices, is: $\boxed{i^i \in \mathbb{R}}$ no matter how you defined it.)

(9.9) It is clear that the domain of z^α can be enlarged if $\alpha = n \in \mathbb{Z}$.

$\Big(n \in \mathbb{Z},\ n \geq 0\ ;\quad z^n = e^{n\ln(z)}.\Big)$

for all z only for $z \in \mathbb{C} \setminus \mathbb{R}_{\leq 0}$

Domain of z^α :
(by our definition)

- $\alpha \in \mathbb{Z}_{\geq 0}$: Domain $= \mathbb{C}$.
- $\alpha \in \mathbb{Z}_{<0}$: Domain $= \mathbb{C}^\times = \mathbb{C} \setminus \{0\}$
- $\alpha \notin \mathbb{Z}$: Domain $= \mathbb{C} - \mathbb{R}_{\leq 0}$.

(9.10) Basic properties of z^α.

- $z^\alpha \cdot z^\beta = z^{\alpha+\beta}$
- $\boxed{\frac{d}{dz} z^\alpha = \frac{\alpha}{z} \cdot z^\alpha}$ and $\boxed{z^\alpha \Big|_{z=1} \quad (= 1^\alpha) = 1}$

Differential equation

Initial condition.