Lecture 10

Point at $\infty$ and stereographic projection.

(10.0) Recall (Lecture 4, Section 4.3, page 3) the definition of limit

$$\lim_{z \to z_0} f(z) = A$$

means

For any given $\varepsilon > 0$, we can find $\delta > 0$, that makes the following statement true:

$$0 < |z - z_0| < \delta \Rightarrow |f(z) - A| < \varepsilon$$

Today we are going to consider the cases when $z_0$ or $A$ or both are "infinity".

(10.1) Let $f(z)$ be a function of a complex variable, defined on an open set $\Omega \subseteq \mathbb{C}$.

Assume $\Omega$ contains all the points of $\mathbb{C}$ outside of a finite disc. Meaning: assume there is a real number $N > 0$ such that

$$\{z \in \mathbb{C} \mid |z| > N\} \subseteq \Omega$$

It is only in this context, that we will define

$$\lim_{z \to \infty} f(z)$$

\[\Omega \text{ must contain this open set } \{ |z| > N \} \]
Intuitively speaking, we want to be able to make \(|z|\) as large as we like, while staying in the domain of \(f\), in any direction.

(10.2) Definition. - By \(\lim_{z \to \infty} f(z) = A\) we mean:

For every \(\varepsilon > 0\), we can find \(R > 0\) such that
\[|z| > R \Rightarrow |f(z) - A| < \varepsilon.\]

**Picture:**

[We must find \(R > 0\), so that \(\{ |z| > R \} \to D(A, \varepsilon)\)]

**Diagram:**

[Given \(\varepsilon > 0\), we want \(f(z)\) to take values in \(D(A, \varepsilon)\)]

Remark. \(\lim_{z \to \infty} f(z) = A\) means nothing more that than

\[\lim_{\frac{1}{z} \to 0} f\left(\frac{1}{z}\right) = A\]

That is, perform the change of variables \(w = \frac{1}{z}\).

Then \(z \to \infty\) means, by definition, \(w \to 0\).
Example: 

(i) \[ \lim_{z \to \infty} \frac{z + 1}{2z + 3} = \lim_{w \to 0} \frac{\frac{1}{w} + 1}{\frac{2}{w} + 3} \]

\[ = \lim_{w \to 0} \frac{1 + w}{2 + 3w} = \frac{1}{2} . \]

(ii) \[ \lim_{z \to \infty} e^z = \lim_{w \to 0} e^{\frac{1}{w}} \text{ does not exist.} \]

[ Recall: \( \lim_{x \to 0^+} e^x = +\infty ; \lim_{x \to 0^-} e^x = 0 . \) ]

In fact \( \lim_{y \to \infty} e^{iy} = \lim_{y \to \infty} (\cos(y) + \sin(y)i) \) also does not exist.

since \( \cos(y) \) (and \( \sin(y) \)) oscillate with period = \( 2\pi \).

(10.4) Analogously, by \[ \lim_{z \to z_0} f(z) = \infty , \text{ we mean} \]

- Either \[ \lim_{z \to z_0} \frac{1}{f(z)} = 0 , \text{ or equivalently:} \]

(i) If \( z_0 \in \mathbb{C} \):

\[ \forall M > 0 , \exists \delta > 0 \text{ so that:} \]

\[ 0 < |z - z_0| < \delta \Rightarrow |f(z)| > M . \]

(ii) If \( z_0 = \infty \):

\[ \forall M > 0 , \exists R > 0 \text{ so that:} \]

\[ |z| > R \Rightarrow |f(z)| > M . \]
(10.5)* The point at infinity – Riemann’s idea.

Consider the three dimensional Cartesian coordinate system with coordinates \((x_1, x_2, x_3)\). Let \(S = \{(a_1, a_2, a_3) \in \mathbb{R}^3 \mid a_1^2 + a_2^2 + a_3^2 = 1\}\) [sphere of radius 1 centered at the origin.]

\[ P = (0, 0, 1). \]

Riemann defined a function (call it \(\rho\) - greek version of \(R\) for Riemann):

\[ \rho : S \setminus P \rightarrow (x_1, x_2) \text{ Plane } (= \mathbb{C}) \]

Given \(A\) a point on \(S\); \(\rho(A)\) = intersection of \((x_1, x_2)\) plane with the line through \(A\) \& \(P\) [See picture above].

(10.6)* Formula for \(\rho\): Let \(A = (a, b, c)\), so \(a^2 + b^2 + c^2 = 1\) \[A \neq P\; \text{i.e.} \; c \neq 1\]

Parametric form of the line through \(A\) and \(P\) is:

\[ \text{[From Calculus III]} \]

*optional
\[ L = \text{Line through } A \text{ & } P = \{ (ta, tb, tc-t+1) : t \in \mathbb{R} \} \]

\[
\begin{align*}
&= P \text{ at } t=0 \\
&= A \text{ at } t=1
\end{align*}
\]

\( \rho(A) \) is found by setting \( tc-t+1 = 0 \); i.e., \( t = \frac{1}{1-c} \)

Thus, as a point in \( \mathbb{R}^2 \),

\[ \rho(A) = \left( \frac{a}{1-c}, \frac{b}{1-c} \right) \]

In spherical coordinates:

\[
\begin{align*}
a &= \cos(\theta) \cos(\phi) \\
b &= \sin(\theta) \sin(\phi) \\
c &= \cos(\phi)
\end{align*}
\]

\( 0 \leq \theta \leq 2\pi; \ 0 < \phi < \pi \)

\[ \rho(\theta, \phi) = \left( \cos(\theta), \frac{\sin(\phi)}{1-\cos(\phi)} \right) \]

Thus:

- for fixed \( \phi \); image of \( P \) lies on the circle of radius \( \frac{\sin(\phi)}{1-\cos(\phi)} \)

\[ \frac{\sin(\phi)}{1-\cos(\phi)} \quad \left[ \begin{array}{c}
\frac{\sin(\phi)}{1-\cos(\phi)} = 0 \text{ at } \phi = \pi. \\
\frac{\sin(\phi)}{1-\cos(\phi)} \rightarrow \infty \text{ as } \phi \rightarrow 0.
\end{array} \right. \]

- for fixed \( \theta \); image of \( P \) = ray shooting off from 0 to \( \infty \) at angle \( \theta \) with \( x_1 \)-axis.