Lecture 10

Point at ∞ and stereographic projection.

(10.0) Recall (Lecture 4, Section 4.3, page 3) the definition of limit

$\boxed{\lim_{z \to z_0} f(z) = A}$ means

> For any given $\varepsilon > 0$, we can find $\delta > 0$, that makes the following statement true:
> $0 < |z - z_0| < \delta \Rightarrow |f(z) - A| < \varepsilon$

Today we are going to consider the cases when z_0 or A or both are "infinity".

(10.1) Let $f(z)$ be a function of a complex variable, defined on an open set $\Omega \subseteq \mathbb{C}$.

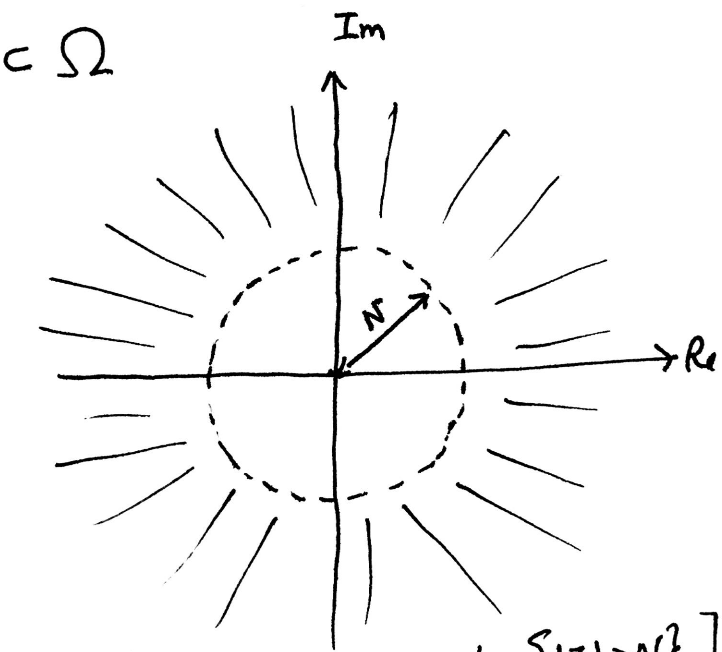
Assume Ω contains all the points of $\mathbb{C}$ outside of a finite disc.

Meaning: assume there is a real number $N > 0$ such that

$$\{z \in \mathbb{C} \mid |z| > N\} \subset \Omega$$

It is <u>only in this context</u>, that we will define

$$\lim_{z \to \infty} f(z)$$



[Ω must contain this open set $\{|z| > N\}$]

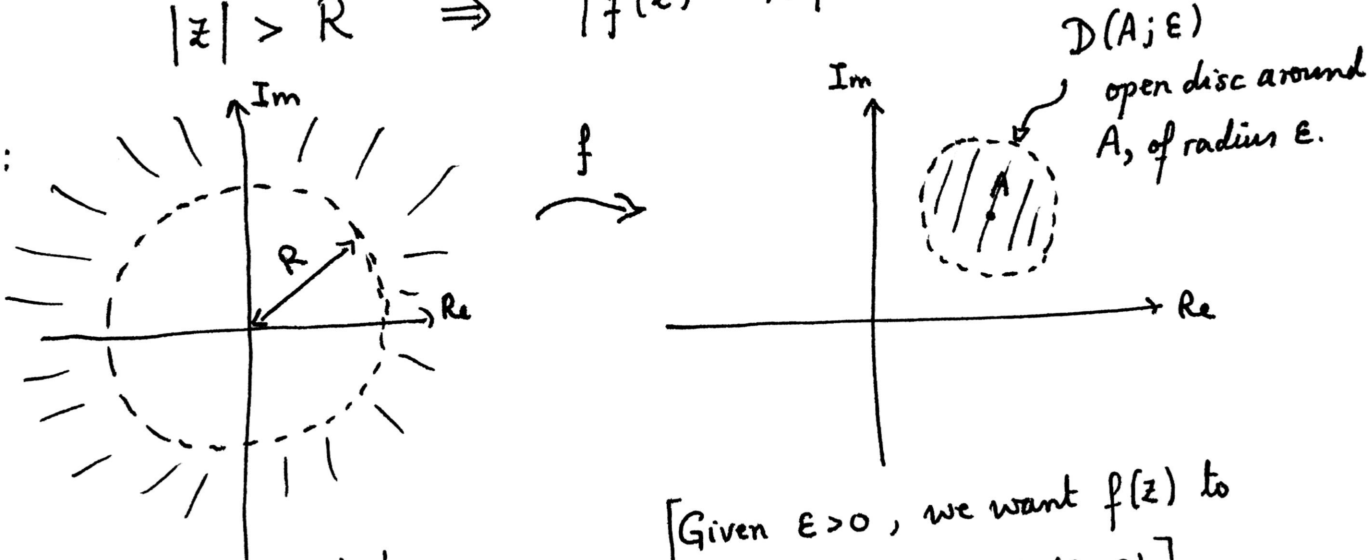
Intuitively speaking, we want to be able to make $|z|$ as large as we like, while staying in the domain of f, in any direction.

(10.2) Definition. – By $\boxed{\lim_{z\to\infty} f(z) = A}$ we mean :

For every $\varepsilon > 0$, we can find $R > 0$ such that

$$|z| > R \quad \Rightarrow \quad |f(z) - A| < \varepsilon .$$

Picture :



[We must find $R>0$, so that $\{|z| > R\} \xrightarrow{f} D(A,\varepsilon)$]

[Given $\varepsilon > 0$, we want $f(z)$ to take values in $D(A,\varepsilon)$]

Remark. $\lim_{z\to\infty} f(z) = A$ means nothing more ~~that~~ than

$$\boxed{\lim_{\frac{1}{z}\to 0} f\left(\tfrac{1}{z}\right) = A}$$

That is, perform the change of variables $w = \frac{1}{z}$.

Then $\boxed{z \to \infty}$ means, by definition, $\boxed{w \to 0}$.

(10.3) Example: (i) $$\lim_{z\to\infty} \frac{z+1}{2z+3} = \lim_{w\to 0} \frac{\frac{1}{w}+1}{\frac{2}{w}+3}$$

$$= \lim_{w\to 0} \frac{1+w}{2+3w} = \frac{1}{2}.$$

(ii) $\displaystyle\lim_{z\to\infty} e^{z} = \lim_{w\to 0} e^{\frac{1}{w}}$ does not exist.

[Recall: $\displaystyle\lim_{\substack{x\to 0^+ \\ (x\in\mathbb{R})}} e^{\frac{1}{x}} = +\infty$; $\displaystyle\lim_{\substack{x\to 0^- \\ (x\in\mathbb{R})}} e^{\frac{1}{x}} = 0$.]

In fact $\displaystyle\lim_{\substack{y\to\infty \\ (y\in\mathbb{R})}} e^{iy} = \lim_{\substack{y\to\infty \\ (y\in\mathbb{R})}} \big(\cos(y)+\sin(y)i\big)$ also does not exist

since $\cos(y)$ (and $\sin(y)$) oscillate with period $= 2\pi$.

(10.4) Analoguously, by $\boxed{\displaystyle\lim_{z\to z_0} f(z) = \infty}$, we mean

here $z_0 \in \mathbb{C}$, or $z_0 = \infty$

- Either $\boxed{\displaystyle\lim_{z\to z_0} \frac{1}{f(z)} = 0}$, or equivalently:

(i) If $z_0 \in \mathbb{C}$: $\forall\, \cancel{\varepsilon>0}\ M>0$, $\exists\, \delta>0$ so that:

$$\boxed{0 < |z-z_0| < \delta \;\Rightarrow\; |f(z)| > M}.$$

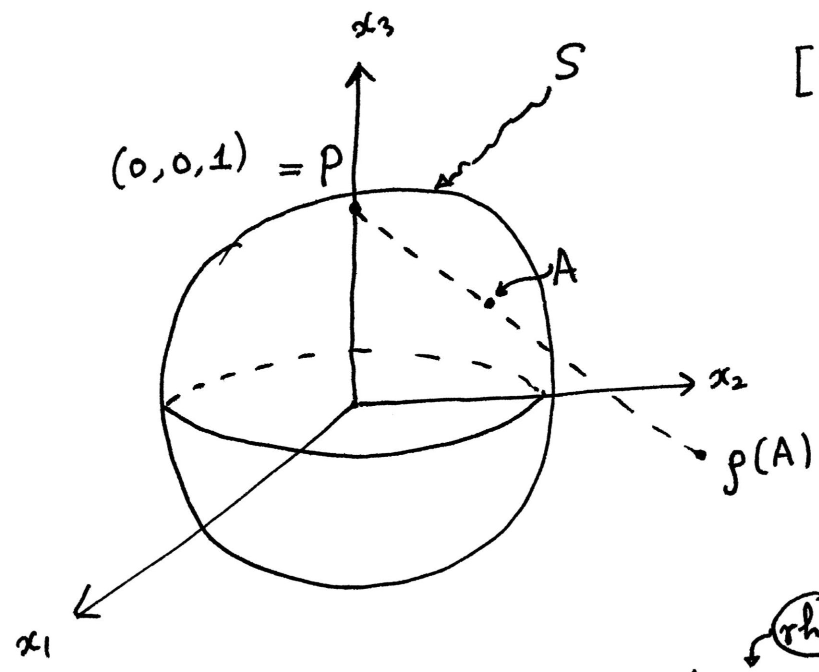
(ii) If $z_0 = \infty$: $\forall\, M>0$, $\exists\, R>0$ so that:

$$\boxed{|z| > R \;\Rightarrow\; |f(z)| > M}.$$

(10.5)* The point at infinity – Riemann's idea.

Consider the three dimensional Cartesian coordinate system with coordinates (x_1, x_2, x_3). Let $S = \{(a_1, a_2, a_3) \in \mathbb{R}^3 \mid a_1^2 + a_2^2 + a_3^2 = 1\}$

[sphere of radius 1 centered at the origin.]



$P = (0,0,1)$.

Riemann defined a function (call it ρ – rho – greek version of R – for Riemann):

$$\rho : S \setminus P \longrightarrow (x_1, x_2) \text{ Plane } (= \mathbb{C})$$

called "stereographic projection"

Given A = a point on S $(A \neq P)$; $\rho(A)$ = intersection of (x_1, x_2) plane with the line through A & P [see picture above].

(10.6)* Formula for ρ. Let $A = (a, b, c)$, so $\boxed{a^2 + b^2 + c^2 = 1}$.
[$A \neq P$; i.e. $c \neq 1$]

Parametric form of the line through A and P is:
[From Calculus III]

* optional

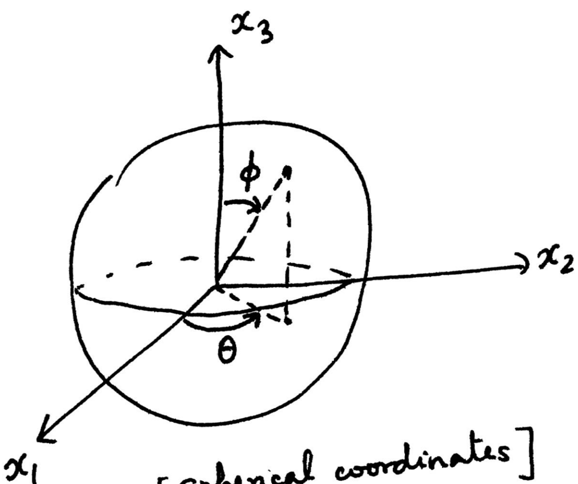
$$\mathcal{L} = \text{Line through } A \,\&\, P = \{ \underbrace{(ta, tb, tc-t+1)}_{\boxed{\substack{= P \text{ at } t=0 \\ = A \text{ at } t=1}}} : t \in \mathbb{R} \}.$$

$\rho(A)$ is found by setting $tc - t + 1 = 0$; i.e., $t = \dfrac{1}{1-c}$

Thus, as a point in $\mathbb{R}^2$, $\boxed{\rho(A) = \left(\dfrac{a}{1-c}, \dfrac{b}{1-c} \right)}$

In spherical coordinates: $\cancel{a = \cos(\theta)\cos\phi}$ $\begin{cases} a = \cos(\theta)\sin(\phi) \\ b = \sin(\theta)\sin(\phi) \\ c = \cos(\phi) \end{cases}$

$0 \leq \theta \leq 2\pi$; $0 < \phi \leq \pi$



[spherical coordinates]

$$\boxed{\rho(\theta, \phi) = \left(\cos(\theta) \cdot \frac{\sin(\phi)}{1-\cos(\phi)}, \sin(\theta) \frac{\sin(\phi)}{1-\cos(\phi)} \right)}$$

Thus: • for fixed ϕ; image of ρ lies on the circle of radius $\dfrac{\sin(\phi)}{1-\cos(\phi)}$. $\left[\dfrac{\sin(\phi)}{1-\cos(\phi)} \begin{array}{l} = 0 \text{ at } \phi = \pi. \\ \to \infty \text{ as } \phi \to 0. \end{array} \right]$

This is >0 for $0 < \phi \leq \pi$

• for fixed θ; image of ρ = ray shooting off from 0 to ∞ at angle θ with x_1-axis