Lecture 11

(11.0) For the next few weeks, we will be dealing with "path integrals" or "contour integrals".

Input : • a $\mathbb{C}$-differentiable function $f : \Omega \to \mathbb{C}$

($\Omega \subseteq \mathbb{C}$ an open, connected subset.)

• $\gamma : [0,1] \longrightarrow \Omega$ a path (i.e. parametric curve)

Output : $\boxed{\int_\gamma f(z)\, dz \quad \in \mathbb{C}}$ - its definition will be given in terms of Riemannian Sums later.

Today we will go over the foundations of the theory of integration as a review of this concept you may have already seen in a previous Calculus course.

(11.1)* Historical context.

(a) The idea of dividing a region into tiny rectangles/squares to guess, roughly, what its area is, goes back to the beginnings of civilization. More concrete description of this idea is often attributed to Greek philosophers (Antiphon ~ 430 BC ; Euclid 322-283 BC ; Archimedes 287-217 BC).

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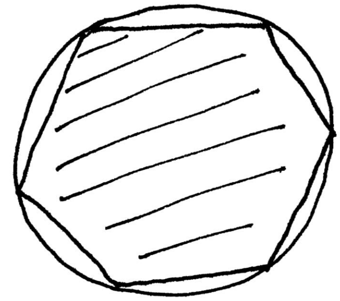
* Optional

They coined the term "Method of exhaustion", by which they meant the following:

If we want to compute area within a circle, we should inscribe a regular polygon in it (say with n-sides), and compute the area of the polygon.



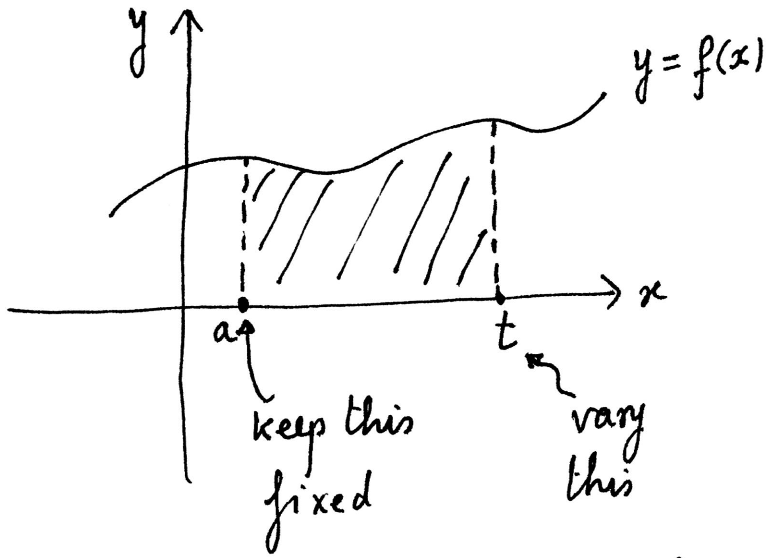
[A hexagon inscribed in a circle.]

As we increase the number of sides, the resulting polygons eventually exhaust the entire circle.

Archimedes used this method to compute areas of many "curved regions" in his book titled "Method".

(b) The Greek method of exhaustion, in one form or the other, was used by scientists of many generations (most notably, Kepler 1571-1630; Galileo 1564-1642). Eventually in the hands of Newton (1642-1727) and Leibniz (1646-1716) it took the form known as "method of quadrature".

They stumbled upon the problem of finding area under the graph of a function, in an attempt to find the primitive (or antiderivative) of this function (now called the fundamental theorem of calculus).



$F(t) :=$ area bounded between $\begin{cases} y = f(x) \\ x\text{-axis between} \\ a \text{ and } t \end{cases}$

Input : $f(x)$. Output (new function) $F(t)$.

$$\boxed{\text{Theorem : } \quad F'(t) = f(t)}.$$

(c) Method of quadrature† (à la Newton & Leibniz).

Given a function $f(x)$ as before, this method (of quadrature) computes the value of the "newly defined" $F(t)$ above, with high degree of precision.

<u>Method</u> : 1. Pick $N \in \mathbb{Z}_{\geq 1}$ and cut the interval $[a, b]$ into N equal pieces.

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† quadrature = "turn into a square"

2. Draw N rectangles inscribed within $y = f(x)$.

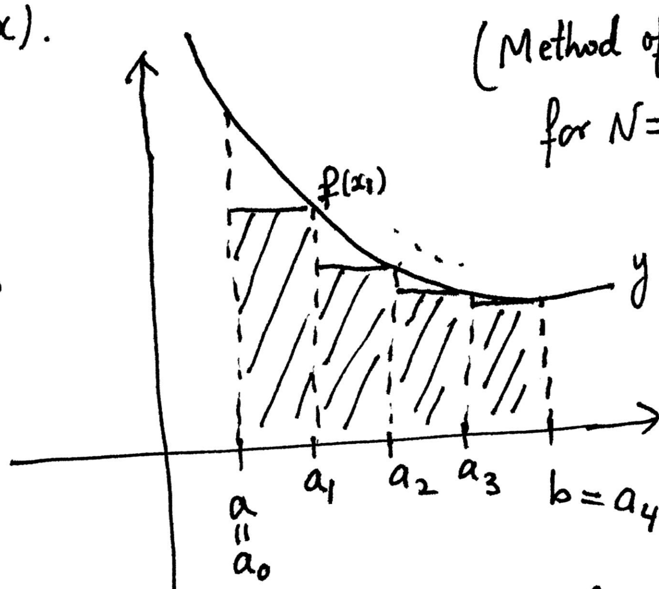
More precisely

- $a_0 = a < a_1 < a_2 \cdots < a_N = b$

  $\left(a_{i+1} - a_i = \dfrac{b-a}{N} . \right)$

- Within each $[a_i, a_{i+1}]$ pick x_i where $f(x_i)$ is smallest.

(Method of quadrature for $N=4$)



$F(b)$ = area bounded between $y = f(x)$, $a \leq x \leq b$

Then $$F(b) \sim \sum_{i=0}^{N-1} \underbrace{f(x_i)}_{\text{height}} \cdot \underbrace{(a_{i+1} - a_i)}_{\text{base (also written as } \Delta x_i)}$$

$$= \frac{b-a}{N} \cdot \sum_{i=0}^{N-1} f(x_i)$$

We can get better and better precision by increasing N.

(d) New notation introduced.

$$F(b) = \int_a^b f(x)\,dx = \lim_{N \to \infty} \sum_{i=0}^{N-1} f(x_i)\,\Delta x_i$$

Notes. – The mathematicians of 18^{th} century (e.g. Euler) perfected this method of quadrature as a powerful tool to

- discover new functions (Slogan of the time: we need more functions to describe physical reality.)
- solve differential equations ("It is very important to solve differential equations" – Newton.)

They never felt to need to define "$\lim\limits_{N \to \infty}$" - a concept that was intuitively clear to them (so why split hair?)

(e) Jean-Baptiste Joseph Fourier (1768-1830) in his work on heat conductivity* claimed a bold statement:

"Every function can be expressed in terms of sine/cosine."

More precisely (not really!): given $f(x)$, Fourier claimed the existence and uniqueness of real numbers

$$c, \ a_1, b_1, \ a_2, b_2, \ \ldots \qquad (\in \mathbb{R})$$

so that
$$f(x) = c + \sum_{n=1}^{\infty} \left(a_n \cos(nx) + b_n \sin(nx) \right).$$

* Théorie analytique de la chaleur (1822)

Stated as above, the assertion is false. The controversy that it stirred made the mathematicians of the time realize that we never bothered to ask

(1) What is a function?

(2) What is a limit?

The first question is answered immediately ($f : A \to B$; - f takes inputs from a set A and gives outputs which are elements of another set B). The second was answered by Cauchy

(f) Augustin-Louis Cauchy (1789-1857).

- $S = \lim_{n \to \infty} s_n$ means: given $\varepsilon > 0$ (how precise you want your answer to be)

there exists $N > 0$ such that

$$|s_n - S| < \varepsilon \quad \text{for every } n \geq N.$$

[The precision can be achieved - after N steps.]

- $$F(t) = \int_a^t f(x)\, dx = \lim_{N \to \infty} \sum_{i=0}^{N-1} f(x_i)\, \Delta x_i$$

has a "meaning" now - and Cauchy carefully proved the fundamental theorem of calculus: $F'(t) = f(t)$.

assuming, of course, that the $\lim_{N\to\infty}$ exists.

The necessary and sufficient conditions for such limit to exist (otherwise said: "f is integrable" (or Riemann integrable).) were worked out by Georg Friedrich Bernhard Riemann (1826–1866).

It is in his honour that we call $\sum_{i=0}^{N-1} f(x_i)\,\Delta x_i$ "Riemann sums" - and functions meeting his conditions "Riemann integrable", or <u>integrable in the sense of Riemann.</u>

(g) $f(x)$ is said to be integrable (from a to b) (Riemannian) if the following holds:

- pick $N > 0$; subdivision $a = a_0 < a_1 < \ldots < a_N = b$
- for each $i = 0, 1, \ldots, N-1$, let $\delta f_i = \left|\begin{array}{l}\text{largest value} \\ f \text{ takes in} \\ [a_i, a_{i+1}]\end{array} - \begin{array}{l}\text{Smallest value } f \\ \text{takes} \\ \text{in } [a_i, a_{i+1}]\end{array}\right|$

(fluctuation of f in $[a_i, a_{i+1}]$)

$$f \text{ is integrable} \iff \sum_{i=0}^{N-1} \delta f_i \cdot (a_{i+1} - a_i) \xrightarrow[\text{as } N\to\infty]{} 0.$$