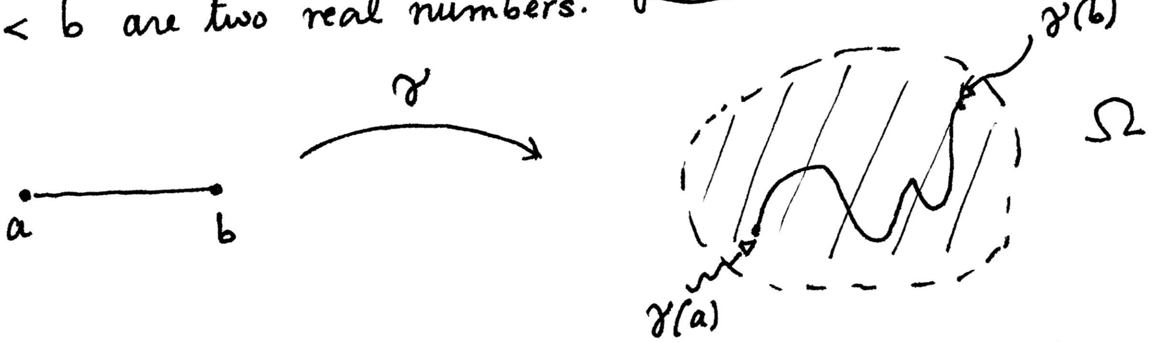
Lecture 12

(12.0) Let $\Omega \subseteq \mathbb{C}$ be an open, connected subset. A path in Ω is a continuous function $\gamma : [a,b] \longrightarrow \Omega$, where $a < b$ are two real numbers.

gamma



In other words, γ is given by 2 real valued functions, $x(t), y(t)$:

$$\gamma(t) = x(t) + y(t)\, i \qquad (a \le t \le b).$$

And, γ being continuous means $x(t)$ and $y(t)$ are continuous.

(12.1) Piecewise smooth paths.

We say γ is smooth if $\gamma'(t) = x'(t) + y'(t)\, i$ exists and is continuous, for every $a < t < b$.

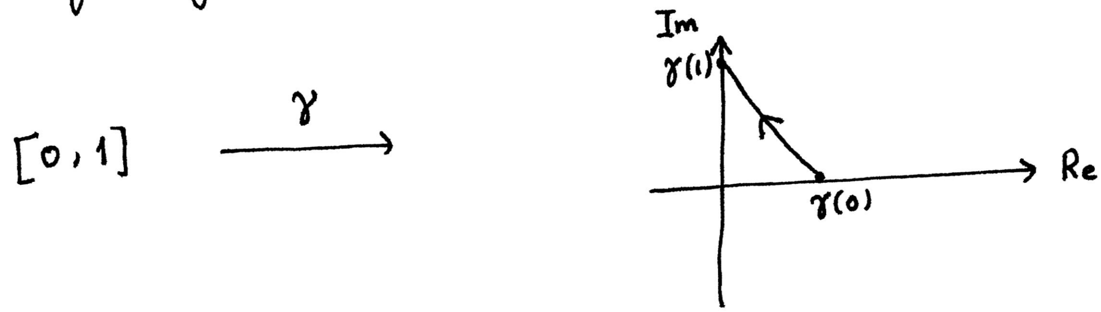
γ is said to be piecewise smooth if it is smooth except at finitely many points in $[a,b]$. Meaning: there are finite numbers $c_1, c_2, \dots, c_\ell$ $(a < c_1 < c_2 \dots < c_\ell < b)$ such that γ is smooth on each subinterval

$$[a, c_1] \ ; \ [c_1, c_2] \ ; \ \dots \ ; \ [c_{\ell-1}, c_\ell] \text{ and } [c_\ell, b]$$

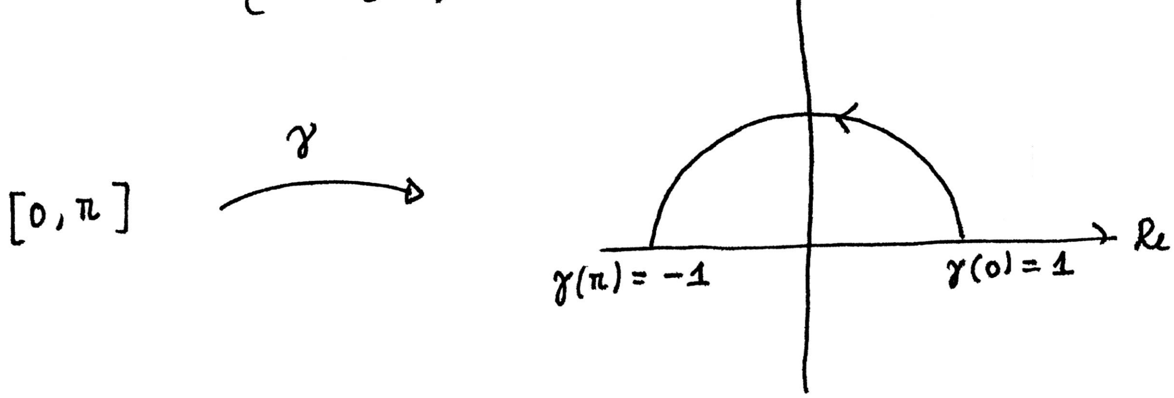
Remark. – Such paths in Ω were called "parametric curves" in Calculus III.

(12.2) Examples. ($\Omega = \mathbb{C}$).

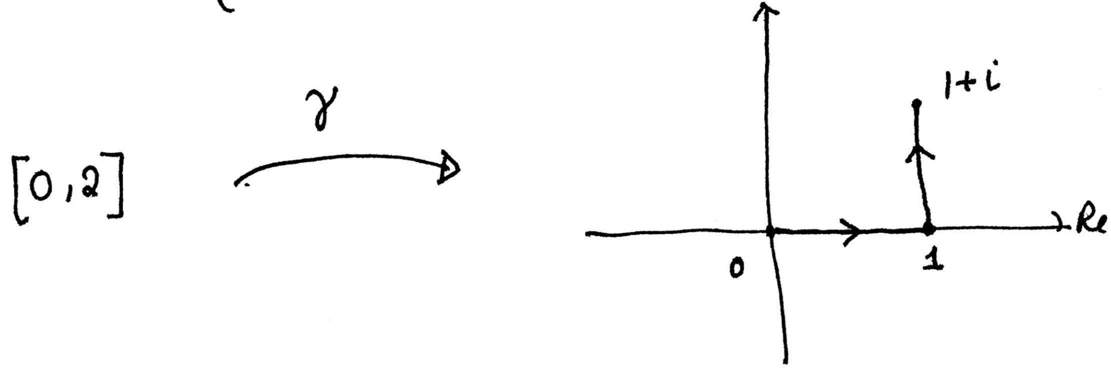
(i) $\gamma(t) = (1-t) + (2t)i$; $0 \le t \le 1$ is a line segment joining $\gamma(0) = 1$ and $\gamma(1) = 2i$



(ii) $\gamma(t) = \cos(t) + \sin(t)\, i$; $0 \le t \le \pi$
$(= e^{it})$



(iii)
$$\gamma(t) = \begin{cases} t & \text{for } 0 \le t \le 1 \\ 1 + (t-1)i & \text{for } 1 \le t \le 2 \end{cases}$$



Examples (i) and (ii) are smooth. (iii) is piecewise smooth.

(12.3) Assume that we are given a (piecewise) smooth path $\gamma : [a,b] \to \Omega$; and a continuous $\mathbb{C}$-valued function $f : \Omega \to \mathbb{C}$. We are going to define:

$$\int_\gamma f(z)\,dz \quad \in \mathbb{C}.$$

Definition 1 (Riemannian Sums).

- Pick $N \in \mathbb{Z}_{\geq 1}$ and subdivide $[a,b]$ into N subintervals.

$$a = t_0 < t_1 < \dots < t_N = b$$

- From each of the subintervals $[t_0, t_1]; [t_1, t_2]; \dots; [t_{N-1}, t_N]$ choose a number:

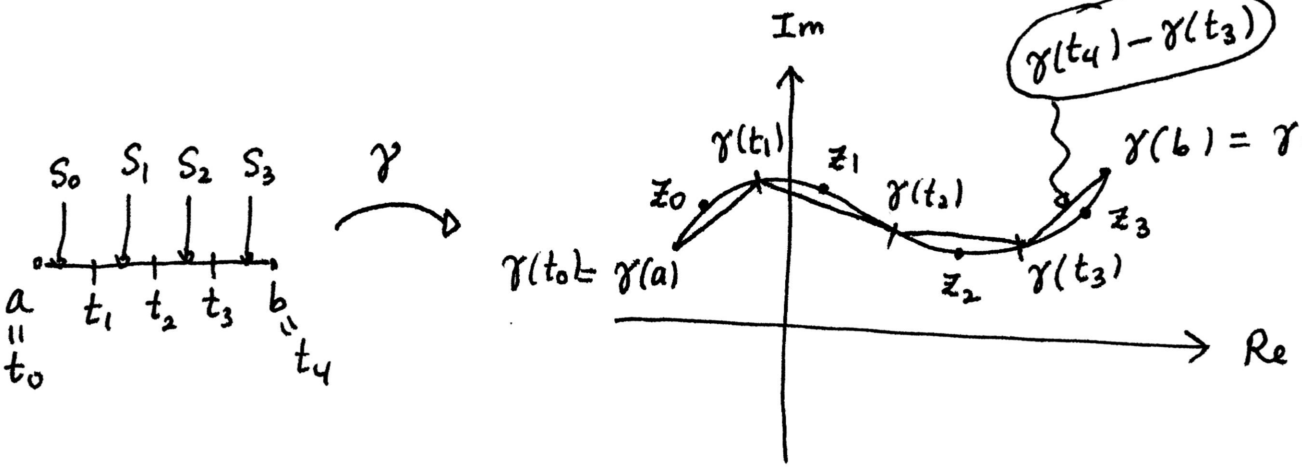
$$s_0 \in [t_0, t_1],\ s_1 \in [t_1, t_2],\ \dots,\ s_{N-1} \in [t_{N-1}, t_N]$$

- Form the sum

$$f(\gamma(s_0)) \cdot (\gamma(t_1) - \gamma(t_0)) + f(\gamma(s_1))(\gamma(t_2) - \gamma(t_1)) + \dots + f(\gamma(s_{N-1}))(\gamma(t_N) - \gamma(t_{N-1}))$$

$$\left(= \sum_{j=0}^{N-1} f(\gamma(s_j)) \underbrace{\;\cdot\;}_{\substack{\text{multiplication}\\ \text{of complex numbers}}} (\gamma(t_{j+1}) \underbrace{-}_{\substack{\text{difference of}\\ \text{complex numbers}}} \gamma(t_j)) \right)$$

$$\int_\gamma f(z)\,dz = \lim_{N\to\infty} \sum_{j=0}^{N-1} f(\gamma(s_j))\cdot(\gamma(t_{j+1})-\gamma(t_j))$$



[illustration : with $N=4$.
z_0, z_1, z_2, z_3 : values of γ at s_0, s_1, s_2, s_3]

[Partial sum =
$f(z_0)\cdot(\gamma(t_1)-\gamma(t_0))$
$+ f(z_1)(\gamma(t_2)-\gamma(t_1))$
$+ f(z_2)(\gamma(t_3)-\gamma(t_2))$
$+ f(z_3)(\gamma(t_4)-\gamma(t_3))$]

<u>Definition 2.</u> (More computationally friendly).

(1) Assume γ is smooth. Then:

$$\boxed{\int_\gamma f(z)\,dz = \int_a^b f(\gamma(t))\cdot\gamma'(t)\,dt}$$

(2) If γ is piecewise smooth, take the definition (1) on each smooth part and add the answers.

Meaning: if $c_1, c_2, \dots, c_\ell$ are the corner points :

$a < c_1 < c_2 < \dots < c_\ell < b$, so that γ is smooth on each subinterval $[a, c_1], [c_1, c_2], \dots, [c_{\ell-1}, c_\ell], [c_\ell, b]$. Then:

$$\int_\gamma f(z)\,dz = \int_a^{c_1} f(\gamma(t))\gamma'(t)\,dt + \int_{c_1}^{c_2} f(\gamma(t))\gamma'(t)\,dt$$

$$+ \dots + \int_{c_\ell}^{b} f(\gamma(t))\gamma'(t)\,dt.$$

(12.4) Remarks - (1)* The two definitions given above are equivalent. The proof of this equivalence requires a little care (is omitted in these notes), but the idea behind the proof can be stated as follows. [Assume γ is smooth.]

As N (from Definition 1) becomes larger and larger, each subinterval $[t_j, t_{j+1}]$ becomes small, and we can replace

$$\frac{\gamma(t_{j+1}) - \gamma(t_j)}{t_{j+1} - t_j} \approx \gamma'(s_j) \quad (s_j \in [t_j, t_{j+1}])$$

and show that

$$\underbrace{\lim_{N\to\infty} \sum_{j=0}^{N-1} f(\gamma(s_j))\,\cancel{t_j}\big(\gamma(t_{j+1}) - \gamma(t_j)\big)}_{\text{Defn 1}} = \underbrace{\lim_{M\to\infty} \sum_{j=0}^{M-1} f(\gamma(s_j))\cdot\gamma'(s_j)\cdot(t_{j+1} - t_j)}_{\text{Defn 2.}}$$

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*Optional

(2) $\boxed{\int_a^b f(\gamma(t))\,\gamma'(t)\,dt}$ is the good old definite integral you have seen in Calculus II:

If $f(z) = u(x,y) + v(x,y)i$; $\gamma(t) = x(t) + y(t)i$;

Then : $$\int_a^b f(\gamma(t))\,\gamma'(t)\,dt = \int_a^b \big(u(x(t),y(t)) + v(x(t),y(t))i\big)\cdot (x'(t)+y'(t)i)\,dt$$

$$= \int_a^b \Big(u(x(t),y(t))\cdot x'(t) - v(x(t),y(t))\cdot y'(t)\Big)\,dt$$

$$+\ i \int_a^b \Big(u(x(t),y(t))\,y'(t) + v(x(t),y(t))\,x'(t)\Big)\,dt$$

e.g. $f(z) = \mathrm{Re}(z)$ (i.e. $u(x,y) = x$ and $v(x,y) = 0$).

$\gamma(t) = \cos(t) + \sin(t)i$; $0 \le t \le \pi$ (example (ii) of (12-2))

Then $$\int_\gamma f(z)\,dz = \int_0^\pi \underbrace{\cos(t)(-\sin(t))}_{\sin(t)\cos(t) = \frac{\sin(2t)}{2}}dt + i\int_0^\pi \underbrace{\cos(t)\cdot\cos(t)}_{\cos^2(t) = \frac{\cos(2t)+1}{2}}\,dt$$

$$= -\frac{1}{2}\int_0^\pi \sin(2t)\,dt + \frac{i}{2}\int_0^\pi (\cos(2t)+1)\,dt$$

$$= -\frac{1}{2}\left[-\frac{\cos(\cancel{2t})}{\cancel{2}}\right]_0^{\pi} + \frac{i}{2}\left[\frac{\sin(\cancel{2t})}{\cancel{2}} + t\right]_0^{\pi}$$

$$0 \qquad\qquad 0$$

$$= \frac{i}{2}\cdot\pi .$$

(12.5) Length of γ is defined to be:

$$\boxed{L(\gamma) = \int_a^b |\gamma'(t)|\, dt}$$

Ex. This definition was also given in Calculus III. Convince yourself that the right-hand side in fact computes the "length" of a parametric curve $\gamma(t)$.

(12.6) Basic properties of $\int_\gamma f(z)\,dz$. (obvious from either of the two definitions (12.3))

(i) $$\int_\gamma (a_1 f_1(z) + a_2 f_2(z))\, dz = a_1 \cdot \int_\gamma f_1(z)\,dz + a_2 \int_\gamma f_2(z)\,dz ,$$

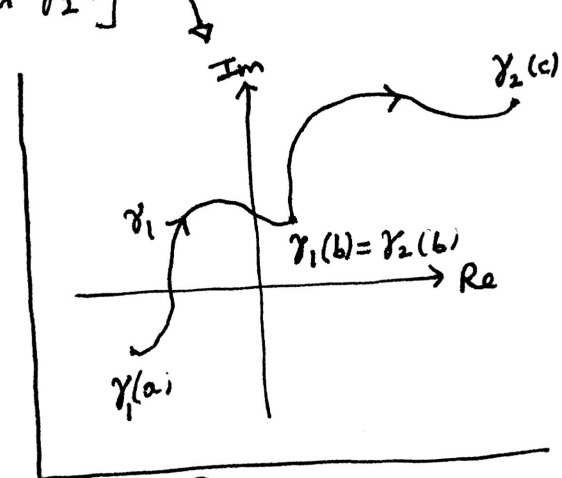
where $a_1, a_2 \in \mathbb{C}$.

(ii) Let $\gamma_1 : [a,b] \longrightarrow \Omega$ and $\gamma_2 : [b,c] \longrightarrow \Omega$

be two (piecewise smooth) paths such that $\gamma_1(b) = \gamma_2(b)$.

Define $\gamma : [a,c] \longrightarrow \Omega$ by: $\gamma(t) = \begin{cases} \gamma_1(t) & \text{if } a \leq t \leq b \\ \gamma_2(t) & \text{if } b \leq t \leq c. \end{cases}$

[γ is obtained by concatenating γ_1 with γ_2.]

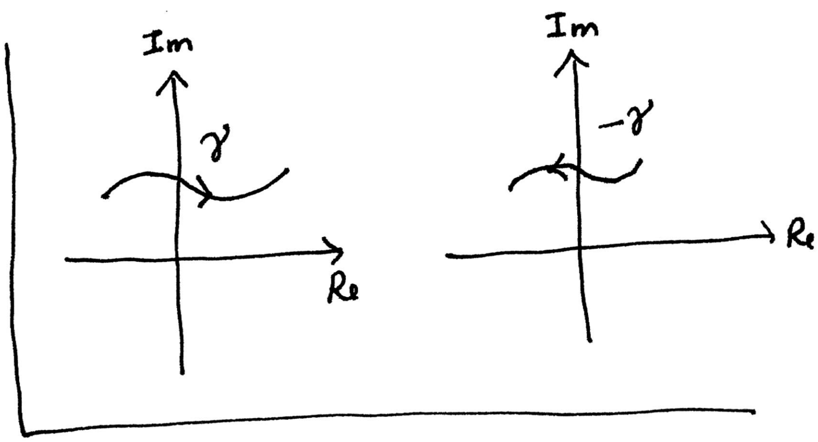


Then $$\int_\gamma f(z)\,dz = \int_{\gamma_1} f(z)\,dz + \int_{\gamma_2} f(z)\,dz.$$

(iii) Given a (piecewise smooth) path $\gamma : [a,b] \longrightarrow \Omega$; define

$-\gamma : [-b,-a] \longrightarrow \Omega$ by $(-\gamma)(t) = \gamma(-t)$.

(same as γ but followed in reverse.)



Then
$$\int_{-\gamma} f(z)\,dz = -\int_\gamma f(z)\,dz$$

(12.7) Important inequality.

$\left[\underset{\text{piecewise smooth path}}{\gamma : [a,b] \to \Omega}\right]$ and $\left[\underset{\text{continuous}}{f : \Omega \to \mathbb{C}}\right]$.

Let $M \in \mathbb{R}_{>0}$ be such that $|f(\gamma(t))| \leq M$ for every $t \in [a,b]$,

and let $L =$ length of γ $\left(= \int_a^b |\gamma'(t)|\,dt \text{ - see (12.5) above.}\right)$

Then $\boxed{\left| \int_\gamma f(z)\,dz \right| \leq M.L}$

Proof I. (Using Riemann Sums). - see notations from Defn 1 of (12.3)

$$\left| \int_\gamma f(z)\,dz \right| = \left| \lim_{N\to\infty} \sum_{j=0}^{N-1} f(\gamma(s_j)) \left(\gamma(t_{j+1}) - \gamma(t_j)\right) \right|$$

$$= \lim_{N\to\infty} \left| \sum_{j=0}^{N-1} f(\gamma(s_j)) \left(\gamma(t_{j+1}) - \gamma(t_j)\right) \right|$$

$$\underset{\text{(triangle ineq.)}}{\leq} \lim_{N\to\infty} \sum_{j=0}^{N-1} \underbrace{|f(\gamma(s_j))|}_{\leq M} . |\gamma(t_{j+1}) - \gamma(t_j)|$$

$$\leq M . \underbrace{\boxed{\lim_{N\to\infty} \sum_{j=0}^{N-1} |\gamma(t_{j+1}) - \gamma(t_j)|}}_{\text{Length of } \gamma} = M.L.$$

□

Proof II. (second defn.)

$$\left| \int_\gamma f(z)\,dz \right| = \left| \int_a^b f(\gamma(t)) \cdot \gamma'(t)\,dt \right|$$

$$\leq \int_a^b \underbrace{|f(\gamma(t))|}_{\leq M} \cdot |\gamma'(t)|\,dt$$

(same logic with Riemann sums & triangle ineq.)

$$\leq M \cdot \underbrace{\boxed{\int_a^b |\gamma'(t)|\,dt}}_{=L} = M \cdot L.$$

$\square$