Lecture 13

(13.0) Set up: • $\Omega \subseteq \mathbb{C}$ open, connected set

• $f : \Omega \longrightarrow \mathbb{C}$ continuous function

• $\gamma : [a,b] \longrightarrow \Omega$ piecewise smooth path.

Recall that we defined (Lecture 12, (12.3) Defn 2):

$$\int_\gamma f(z)\,dz = \int_a^b f(\gamma(t))\,\gamma'(t)\,dt \quad (\text{if } \gamma \text{ is smooth})$$

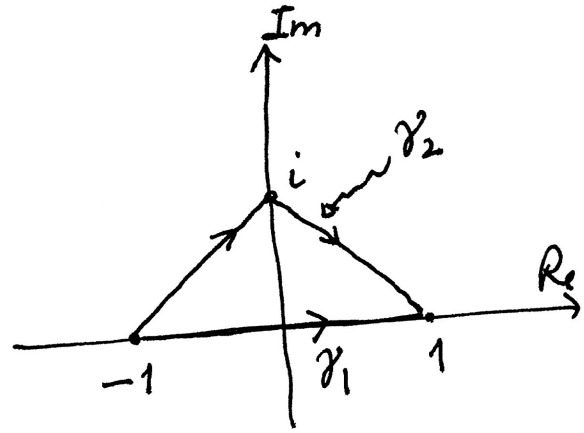
$$= \int_a^{c_1} f(\gamma(t))\,\gamma'(t)\,dt + \int_{c_1}^{c_2} f(\gamma(t))\,\gamma'(t)\,dt + \cdots + \int_{c_\ell}^{b} f(\gamma(t))\,\gamma'(t)\,dt$$

(if γ is piecewise smooth - smooth on each $[a,c_1], [c_1,c_2], \ldots, [c_\ell, b]$).

(13.1) Three examples.

I. $f(z) = \mathrm{Re}(z)$. $\gamma_1(t) = t$; $(-1 \leq t \leq 1)$ $(\Omega = \mathbb{C}.)$

$$\gamma_2(t) = \begin{cases} (-1+t) + t i \;; & 0 \leq t \leq 1 \\ (-1+t) + (2-t) i \;; & 1 \leq t \leq 2 \end{cases}$$



$$\int_{\gamma_1} f(z)\,dz = \int_{-1}^{1} \underbrace{t}_{f(\gamma_1(t))} \cdot \underbrace{1}_{\gamma_1'(t)} \cdot dt = \left[\frac{t^2}{2}\right]_{-1}^{1} = 0.$$

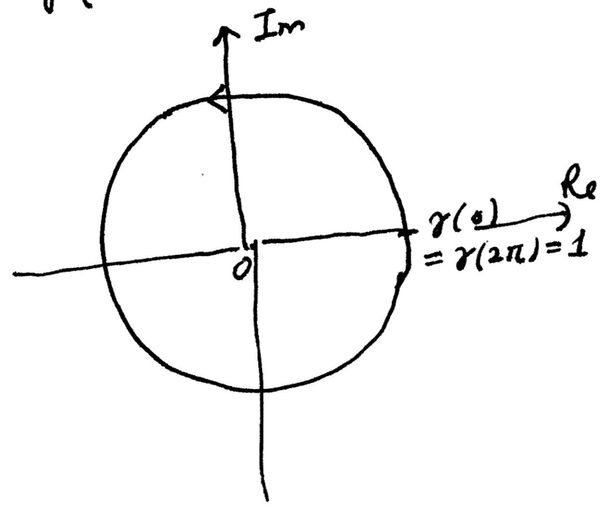
$$\int_{\gamma_2} f(z)dz = \int_0^1 \underbrace{(-1+t)}_{f(\gamma_2(t))} \underbrace{(1+i)}_{\substack{\gamma_2'(t) \\ (\text{for } 0 \leq t \leq 1)}} dt + \int_1^2 (-1+t)\cdot(1-i)\, dt$$

$$= (1+i) \left[-t+\frac{t^2}{2}\right]_0^1 + (1-i) \left[-t+\frac{t^2}{2}\right]_1^2$$

$$= (1+i)\left(-\frac{1}{2}-0\right) + (1-i)\left(0-\left(-\frac{1}{2}\right)\right) = -i$$

(Moral of the story: the answer depends on how we go from -1 to 1. Reason (as we will see later): $f(z) = \mathrm{Re}(z)$ is NOT $\mathbb{C}$-differentiable.)

II. $f(z) = \frac{1}{z}$. $\Omega = \mathbb{C} \setminus \{0\}$. $\gamma(t) = e^{it}$ $(0 \leq t \leq 2\pi)$

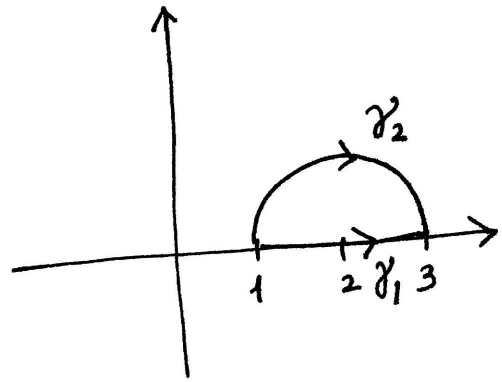


$$\int_\gamma f(z)\,dz = \int_0^{2\pi} \frac{1}{e^{it}} \cdot \underbrace{i \cdot e^{it}}_{\gamma'(t)} dt$$

$$= i \cdot \int_0^{2\pi} 1 \cdot dt = 2\pi i.$$

Moral of the story. - the answer still depends on the path chosen $\left(\begin{array}{l} \text{for } \gamma(t) = e^{it} \ (0 \le t \le 2\pi) \text{ - we get } 2\pi i \\ \text{for } \gamma_2(t) = 1 \ (0 \le t \le 2\pi) \text{ - we get } 0. \end{array} \right)$, even when $f(z)$ is $\mathbb{C}$-differentiable. Reason (again as we will see later) the whole interior of γ is not in the domain of f.

III. $f(z) = z \,;\ \Omega = \mathbb{C}$.



$\gamma_1(t) = t \,;\ 1 \le t \le 3.$

$\gamma_2(t) = 2 - e^{-it} \,;\ 0 \le t \le \pi.$

$$\int_{\gamma_1} f(z)\,dz = \int_1^3 t \cdot 1 \cdot dt = \left[\frac{t^2}{2}\right]_1^3 = 4.$$

$$\int_{\gamma_2} f(z)\,dz = \int_0^\pi \underbrace{(2 - e^{-it})}_{f(\gamma_2(t))} \underbrace{(-1)(-i)e^{-it}}_{\gamma_2'(t)}\,dt$$

$$= i\left[\int_0^\pi 2e^{-it}\,dt - \int_0^\pi e^{-2it}\,dt\right]$$

$$= 2 \cdot i \cdot \left[\frac{e^{-it}}{-i}\right]_0^\pi - i \cdot \underbrace{\left[\frac{e^{-2it}}{-2i}\right]_0^\pi}_{0} = -2(-1-1) = 4.$$

(13.2) Theorem. Let $f : \Omega \to \mathbb{C}$ be a continuous function, where $\Omega \subseteq \mathbb{C}$ is an open and connected subset. Assume that there exists a $\mathbb{C}$-differentiable $F : \Omega \to \mathbb{C}$ such that

$$F'(z) = f(z). \quad \text{(i.e., } F \text{ is } \underline{\text{an}} \text{ antiderivative of } f \text{ on } \Omega.)$$

Then for any piecewise smooth path $\gamma : [a,b] \to \Omega$, we have:

$$\boxed{\int_\gamma f(z)\, dz = F(\gamma(b)) - F(\gamma(a))}$$

(this means that the answer only depends on the starting and finishing points of γ - and not on γ itself. - see Ex. III of (13.1) above.)

Proof. - Let us write $f(z) = u(x,y) + v(x,y)i$,

$$F(z) = U(x,y) + V(x,y)i, \quad \gamma(t) = x(t) + y(t)i.$$

→ Assume that γ is smooth. Then:

(i) $\boxed{F(z) \text{ is } \mathbb{C}\text{-differentiable and } F'(z) = f(z)}$ means

$$U_x = V_y = u$$
$$V_x = -U_y = v$$

(Cauchy-Riemann)

(ii) Unfolding $\int_a^b f(\gamma(t))\,\gamma'(t)\,dt$ (see Lecture 12, §12.4, Remark (2))

we get:

- $$\mathrm{Re}\left(\int_\gamma f(z)\,dz\right) = \int_a^b \Big(u(x(t),y(t))\cdot x'(t) - v(x(t),y(t))\cdot y'(t)\Big)\,dt$$

$$= \int_a^b \Big(\underbrace{U_x(x(t),y(t))\cdot y'(t) + U_y(x(t),y(t))\cdot y'(t)}_{\frac{d}{dt}\left(U(x(t),y(t))\right) \text{ by Chain rule.}}\Big)\,dt$$

$$= \Big[U(x(t),y(t))\Big]_a^b = \mathrm{Re}\big(F(\gamma(b)) - F(\gamma(a))\big).$$

- $$\mathrm{Im}\left(\int_a^b f(z)\,dz\right) = \int_a^b \Big(u(x(t),y(t))\,y'(t) + v(x(t),y(t))\,x'(t)\Big)\,dt$$

$$= \int_a^b \Big(\underbrace{V_y(x(t),y(t))\,y'(t) + V_x(x(t),y(t))\,x'(t)}_{\frac{d}{dt}\left(V(x(t),y(t))\right)}\Big)\,dt$$

$$= \Big[V(x(t),y(t))\Big]_a^b$$

$$= \mathrm{Im}\big(F(\gamma(b)) - F(\gamma(a))\big).$$

Hence, $$\int_\gamma f(z)\,dz = F(\gamma(b)) - F(\gamma(a)) \text{ for smooth } \gamma.$$

→ If γ is only piecewise smooth, say smooth on $[a, c_1], [c_1, c_2]$, $\dots, [c_\ell, b]$, then (by the previous argument):

$$\int_\gamma f(z)\,dz = \Big(F(\gamma(c_1)) - F(\gamma(a))\Big) + \Big(F(\gamma(c_2)) - F(\gamma(c_1))\Big)$$
$$+ \dots + \Big(F(\gamma(c_\ell)) - F(\gamma(c_{\ell-1}))\Big) + \Big(F(\gamma(b)) - F(\gamma(c_\ell))\Big)$$
$$= F(\gamma(b)) - F(\gamma(a)). \qquad \square$$

(13.3) Remarks. – (1) In view of this theorem, and Example II of (13.1) above, we conclude that $\frac{1}{z}$ $(\Omega = \mathbb{C} \setminus \{0\})$ does not have an antiderivative on the whole of Ω. This is consistent with the fact that we could not define a logarithm on $\mathbb{C} \setminus \{0\}$, but only on $\mathbb{C} \setminus \mathbb{R}_{\leq 0}$ (Lecture 9; (9.2); page 3).

(2) This theorem will allow us to compute $\int_\gamma f(z)\,dz$ for a lot of functions whose antiderivative is easy to write down: e.g.

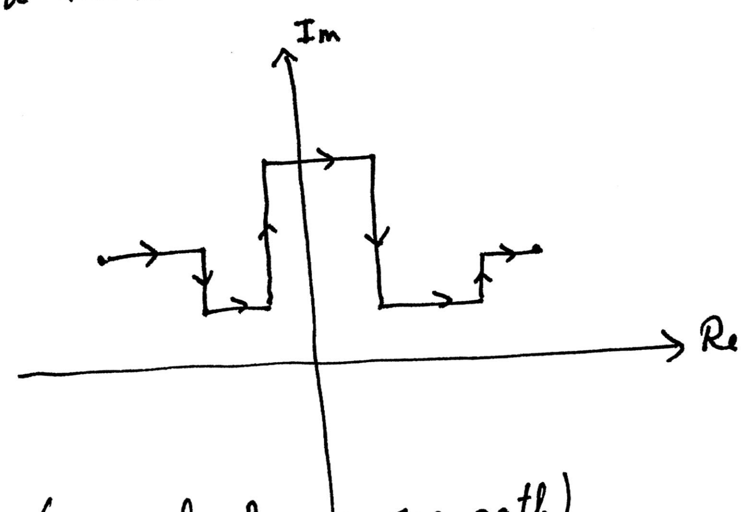
- $f(z) = a_0 + a_1 z + \dots + a_n z^n$; $F(z) = a_0 z + a_1 \frac{z^2}{2} + \dots + a_n \frac{z^{n+1}}{n+1}$
  polynomial; $\Omega = \mathbb{C}$

(13.4) Converse of Theorem (13.2) above is also true. Namely:

> If $f : \Omega \to \mathbb{C}$ is continuous; ($\Omega \subseteq \mathbb{C}$ open & connected)
> such that $\displaystyle\int_{\gamma_1} f(z)\,dz = \int_{\gamma_2} f(z)\,dz$ for any two piecewise smooth paths in Ω, with same end points
>
> Then there exists a $\mathbb{C}$-differentiable $F : \Omega \to \mathbb{C}$ such that
> $$F'(z) = f(z).$$

We are going to need a stronger version of this, which we will prove later, where we only consider <u>zig-zag paths</u>.

(13.5) Zig-zag path. Let $\gamma : [a,b] \longrightarrow \Omega$ be a piecewise smooth path. We say γ is a zig-zag path if every smooth component of γ is either a horizontal line; or a vertical line.



(example of a zig-zag path)

More precisely, $[a,b] = [a,c_1] \cup [c_1,c_2] \cup \dots \cup [c_\ell, b]$

$$\Big(\underset{\substack{\| \\ c_0}}{a} < c_1 < c_2 < \dots < c_\ell < \underset{\substack{\| \\ c_{\ell+1}}}{b} \text{ as before.} \Big)$$

and for each subinterval

$$[c_j, c_{j+1}] \qquad (0 \le j \le \ell \,;\; c_0 = a,\ c_{\ell+1} = b)$$

either $\quad \mathrm{Re}(\gamma(t)) = \text{constant} \quad \forall\, t \in [c_j, c_{j+1}]$ (vertical)

or $\quad \mathrm{Im}(\gamma(t)) = \text{constant} \quad \forall\, t \in [c_j, c_{j+1}]$ (horizontal)

Fact: (I will write a proof of this separately in "Optional Reading" material)

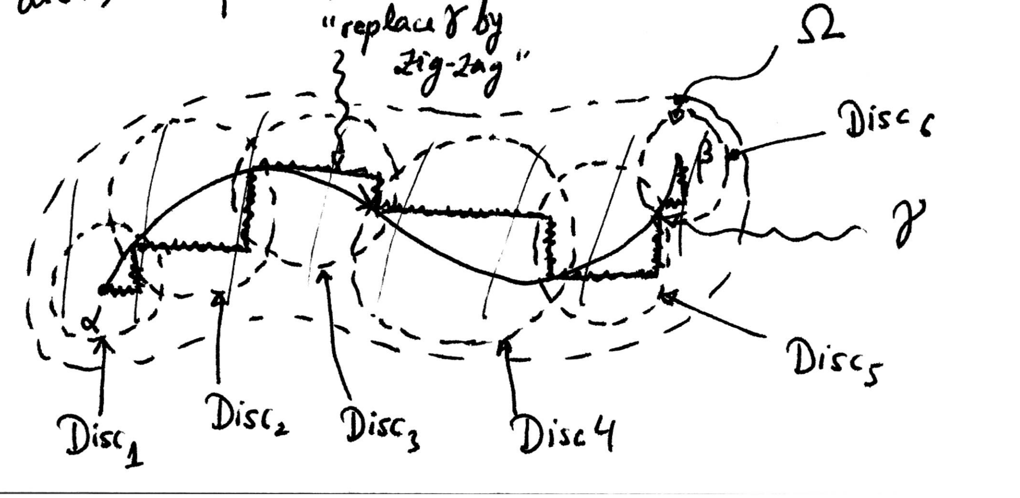
If $\Omega \subseteq \mathbb{C}$ is open and connected, then any two points can be joined by a zig-zag path.

Idea of the proof: Let $\alpha, \beta \in \Omega$. Since Ω is connected, there is some path $\gamma : [a,b] \to \Omega$ with $\gamma(a) = \alpha$, $\gamma(b) = \beta$.

$\begin{bmatrix} \Omega : \text{open} \\ \gamma \text{ closed \& bounded} \end{bmatrix} \Rightarrow$ we can cover γ by a finite number of (open) discs.

Within each open disc, the part of γ can be replaced by a zig-zag

Illustration:



(13.6) We will prove the following theorem next time.

<u>Theorem.</u> Let $\Omega \subseteq \mathbb{C}$ be open and connected subset and $f : \Omega \to \mathbb{C}$ a continuous function. Then, the following three assertions are equivalent.

(1) There exists a $\mathbb{C}$-differentiable function $F : \Omega \to \mathbb{C}$ such that $F'(z) = f(z)$.

(2) Given any two piecewise smooth paths $\gamma_1, \gamma_2 : [a,b] \to \Omega$ such that $\gamma_1(a) = \gamma_2(a)$ and $\gamma_1(b) = \gamma_2(b)$;

$$\int_{\gamma_1} f(z)\,dz = \int_{\gamma_2} f(z)\,dz$$

(3) – same statement as (2) – but γ_1 and γ_2 are only assumed to be zig-zag paths.