Lecture 14

(14.0) Recall: 
- $\Omega \subseteq \mathbb{C}$ is an open and connected subset.
- $f : \Omega \to \mathbb{C}$ a continuous function.

For a smooth $\gamma : [a, b] \to \Omega$, we defined
\[
\int_{\gamma} f(z) \, dz = \int_{a}^{b} f(\gamma(t)) \cdot \gamma'(t) \, dt.
\]

(When $\gamma$ is only piecewise smooth, $\int_{\gamma} f(z) \, dz$ is the sum of these integrals over smooth parts of $\gamma$—see (12.3) or (13.0)).

Last time we introduced \underline{zig-zag paths}: piecewise smooth path whose each smooth segment is either a horizontal line or a vertical line.

We stated the following theorem (see (13.6) — page 9 of Lecture 13).

\underline{Theorem:} The following three statements are equivalent.

1. $f$ admits an antiderivative. Meaning: there exists a $C^1$-differentiable function $F : \Omega \to \mathbb{C}$ so that $F'(z) = f(z)$.

2. $\int_{\gamma} f(z) \, dz$ only depends on the starting and ending points of $\gamma$; for any \underline{piecewise smooth path} $\gamma : [a, b] \to \Omega$.

(\text{More precisely, for any two piecewise smooth paths } \gamma_1, \gamma_2 : [a, b] \to \Omega, \text{ such that } \gamma_1(a) = \gamma_2(a) \text{ and } \gamma_1(b) = \gamma_2(b), \text{ we have: } \int_{\gamma_1} f(z) \, dz = \int_{\gamma_2} f(z) \, dz.)
(3) For any two zig-zag paths $\gamma_1, \gamma_2 : [a, b] \to \Omega$ such that $\gamma_1(a) = \gamma_2(a)$ and $\gamma_1(b) = \gamma_2(b)$, we have

$$\int_{\gamma_1} f(z) \, dz = \int_{\gamma_2} f(z) \, dz.$$

We also showed that (1) $\Rightarrow$ (2) (see Theorem (13.2) - page 4 of Lecture 13).

Since (3) is a special case of (2), (2) $\Rightarrow$ (3).

(14.1) Proof of (3) $\Rightarrow$ (1).*

Pick a point, say $\alpha$, in $\Omega$.

We define $F : \Omega \to \mathbb{C}$ by the following

(i) Given $z_0 \in \Omega$, choose a zig-zag path $\gamma : [0, 1] \to \Omega$

such that $\gamma(0) = \alpha$, $\gamma(1) = z_0$.

(Note: since $\Omega$ is connected, we saw last time that any two points can be joined by a zig-zag path - see (13.5) page 8 of Lecture 13.)

(ii) Define

$$F(z_0) = \int_{\gamma} f(z) \, dz$$

* based on an argument due to Giacinto Morera (1856-1909, Novara, Italy)
Note: This idea is almost verbatim to the one due to Newton and Leibniz (compare with Lecture 11- page 3). The only difference is that, over $\mathbb{R}$ there is only one way to go from $a \in \mathbb{R}$ to $t \in \mathbb{R}$, and over $\mathbb{C}$ there are many ways. (fixed) (varies)

→ The hypothesis from (3) of our theorem ensures that the definition given above
\[
\int_{\gamma} f(z) \, dz
\]
does not depend on which zig-zag path $\gamma$ we chose.

Now we will prove that $F: \Omega \to \mathbb{C}$, thus defined, is $\mathbb{C}$-differentiable, and $F'(z) = f(z)$. By Theorem (6.1) - page 2 of Lecture 6 (Cauchy-Riemann) - it is sufficient to show that:

\[\text{if we write } F(z) = U(x,y) + V(x,y)i \quad ; \quad x = \text{Re}(z), y = \text{Im}(z).\]

\[f(z) = U(x,y) + V(x,y)i\]

(a) $F_x = U_x + V_x i = f$

(b) $-iF_y = V_y - U_y i = f$

[Note: once we prove (a) and (b), we will automatically obtain the continuity of $U_x, U_y, V_x, V_y$, since $f$ is assumed to be continuous - hypotheses (i), (ii), (iii) of Theorem (6.1) will be satisfied.]
Verification of (a): \[ F_x = f \]. Let \( z_0 \in \Omega \).

Then \( F_x(z_0) = \lim_{h \to 0} \frac{F(z_0+h) - F(z_0)}{h} \) \((h \in \mathbb{R})\).

\[ F(z_0) = \int_{\gamma} f(z) \, dz \quad \text{and} \quad F(z_0 + h) = \int_{\gamma} f(z) \, dz + \int_{\gamma_H} f(z) \, dz \]

\[ \gamma_H \text{ is horizontal line joining } z_0 \text{ and } z_0 + h, \text{ e.g., } \gamma_H(t) = z_0 + th; \ 0 \leq t \leq 1 \]

Thus,
\[
\lim_{h \to 0} \frac{F(z_0+h) - F(z_0)}{h} = \lim_{h \to 0} \frac{1}{h} \int_{\gamma_H} f(z) \, dz \\
= \lim_{h \to 0} \frac{1}{h} \int_0^1 f(z_0 + th) \cdot h \cdot dt = \lim_{h \to 0} \int_0^1 f(z_0 + th) \, dt \\
= 1 \\
\]

Claim: \[ \lim_{h \to 0} \int_0^1 f(z_0 + th) \, dt = f(z_0) \] (Hence, (a) follows).
Proof of the claim. By definition of \( \lim_{h \to 0} \), we need to show that: Given \( \epsilon > 0 \), we can find \( \delta > 0 \) so that

\[
0 < |h| < \delta \implies \left| \int_0^1 f(z_0 + th) \, dt - f(z_0) \right| < \epsilon.
\]

Since \( f \) is continuous (at \( z_0 \)), there certainly is some \( \delta > 0 \) so that

\[
0 < |h| < \delta \implies |f(z_0 + k) - f(z_0)| < \epsilon. \quad (k = th).
\]

This \( \delta \) works for us! as shown below:

If \( 0 < |h| < \delta \), then \( |th| < \delta \) and we get:

\[
0 < t < 1
\]

\[
\left| \int_0^1 f(z_0 + th) \, dt - f(z_0) \right| = \left| \int_0^1 (f(z_0 + th) - f(z_0)) \, dt \right|
\]

\[
\leq \int_0^1 |f(z_0 + th) - f(z_0)| \, dt < \int_0^1 \epsilon \, dt = \epsilon.
\]

Verification of (b) is exactly the same, just replace \( y_x \) by \( y_y \).

The theorem is proved.
(14.2) Closed paths, simple paths, contours.

Let \( \gamma : [a, b] \to \mathbb{C} \) be a piecewise smooth path.

We say \( \gamma \) is closed if \( \gamma(a) = \gamma(b) \). We say \( \gamma \) is simple if it does not cross itself, except possibly at end points. That is, for every \( t_1, t_2 : a < t_1 < t_2 < b \); \( \gamma(t_1) \neq \gamma(t_2) \) for every \( t : a < t < b \); \( \gamma(t) \neq \gamma(a) \) and \( \gamma(t) \neq \gamma(b) \).

Contour = simple and closed.

(14.3) Interior, exterior and orientation.

Let \( \gamma : [a, b] \to \mathbb{C} \) be a contour (i.e., \( \gamma \) is simple and closed). Then \( \mathbb{C} \setminus \text{image of } \gamma \) \( (= \{ z \in \mathbb{C} \mid z \neq \gamma(t) \text{ for any } a \leq t \leq b \}) \) has two connected components - a bounded one and an unbounded one.
The bounded component of \( \mathbb{C} \setminus \text{(image of } \gamma \text{)} \) is called the interior of \( \gamma \).

The orientation of \( \gamma \) is said to be positive (or counterclockwise) if the interior of \( \gamma \) is to the left, while traversing along \( \gamma \).

(14.4) Restating our theorem (14.0) above:

- \( \Omega \subseteq \mathbb{C} \) open, connected
- \( f : \Omega \to \mathbb{C} \) continuous

TFAE (acronym for "The following are equivalent")

1. \( f \) admits an antiderivative on \( \Omega \).
2. For every contour \( \gamma : [a, b] \to \Omega \), \( \int_{\gamma} f(z) \, dz = 0 \).
3. For every rectangular contour \( \gamma : [a, b] \to \Omega \), \( \int_{\gamma} f(z) \, dz = 0 \).
Reason why this is merely a restatement:

- given \( \gamma_1, \gamma_2 \) two paths in \( \Omega \) with same endpoints:
  \[ \{ \gamma_1 \text{ concatenated with } (-\gamma_2) \} = \gamma \text{ is a closed path.} \]

\[
\int_{\gamma} f(z) \, dz = \int_{\gamma_1} f(z) \, dz + \int_{(-\gamma_2)} f(z) \, dz = \int_{\gamma_1} f(z) \, dz - \int_{\gamma_2} f(z) \, dz.
\]

(even though \( \gamma \) may not be simple, we can break it into simple pieces.)

\[
\begin{align*}
\alpha \quad & \quad \gamma_1 \quad \beta \\
\rightarrow & \quad \rightarrow \\
\gamma & = \gamma_1 \text{ concatenate } (-\gamma_2) \\
\end{align*}
\]

- Similarly if \( \gamma_1 \) and \( \gamma_2 \) are two zig-zag paths joining \( \alpha \) & \( \beta \);
  \[ \gamma_1 (-\gamma_2) = \gamma \text{ can be written as sum of rectangular paths.} \]
Next time we will prove Cauchy's theorem:

- $\Omega \subseteq \mathbb{C}$ open, connected.
- $f : \Omega \to \mathbb{C}$ differentiable.
- $\gamma$ a contour in $\Omega$ such that $\text{interior}(\gamma) \subseteq \Omega$.

Then $\int_{\gamma} f(z) \, dz = 0$.

This theorem signifies the importance of those subsets $\Omega$ for which the last hypothesis is always true ($\text{interior}(\gamma) \subseteq \Omega$). These are called simply-connected.

A subset $\Omega \subseteq \mathbb{C}$ is called simply-connected if for every contour $\gamma : [a,b] \to \Omega$, we have $\text{interior}(\gamma) \subseteq \Omega$.

$\Omega = D(0,R) = \{ |z| < R \}$ is simply-connected.

$D(0,R) = \{ 0 < |z| < R \}$ is not simply connected.