Lecture 14

(14.0) Recall:
- $\Omega \subseteq \mathbb{C}$ is an open and connected subset.
- $f : \Omega \to \mathbb{C}$ a continuous function.

For a smooth $\gamma : [a,b] \to \Omega$, we defined

$$\int_\gamma f(z)\,dz = \int_a^b f(\gamma(t)) \cdot \gamma'(t) \cdot dt .$$

(When γ is only piecewise smooth, $\int_\gamma f(z)\,dz$ is the sum of these integrals over smooth parts of γ - see (12.3) or (13.0).)

Last time we introduced zig-zag paths: piecewise smooth path whose each smooth segment is either a horizontal line or a vertical line.

We stated the following theorem (see (13.6) - page 9 of Lecture 13).

Theorem: The following three statements are equivalent.

(1) f admits an antiderivative. Meaning: there exists a $\mathbb{C}$-differentiable function $F : \Omega \to \mathbb{C}$ so that $\boxed{F'(z) = f(z)}$.

(2) $\int_\gamma f(z)\,dz$ only depends on the starting and ending points of γ; for any piecewise smooth path $\gamma : [a,b] \to \Omega$.

(More precisely, for any two piecewise smooth paths $\gamma_1, \gamma_2 : [a,b] \to \Omega$, such that $\gamma_1(a) = \gamma_2(a)$ and $\gamma_1(b) = \gamma_2(b)$, we have:

$$\int_{\gamma_1} f(z)\,dz = \int_{\gamma_2} f(z)\,dz .)$$

(3) For any two <u>zig-zag paths</u> $\gamma_1, \gamma_2 : [a,b] \to \Omega$ such that $\gamma_1(a) = \gamma_2(a)$ and $\gamma_1(b) = \gamma_2(b)$, we have

$$\int_{\gamma_1} f(z)\, dz = \int_{\gamma_2} f(z)\, dz .$$

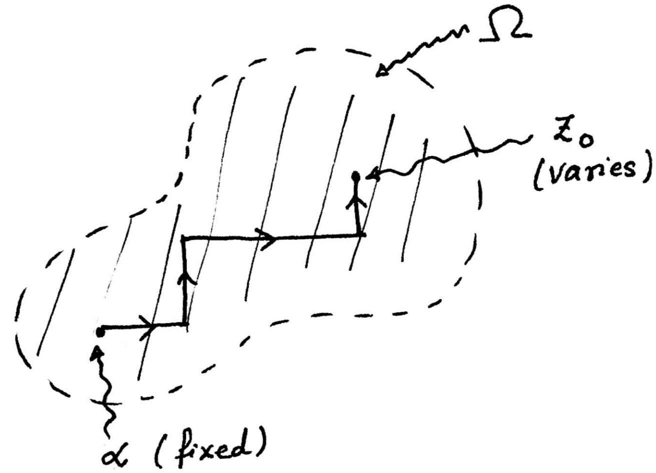
We also showed that (1) $\Rightarrow$ (2) (see Theorem (13.2) - page 4 of Lecture 13).

Since (3) is a special case of (2), (2) $\Rightarrow$ (3).

(14.1) Proof of (3) $\Rightarrow$ (1).*

Pick a point, say α, in Ω.

We define $F : \Omega \to \mathbb{C}$ by the following



(i) Given $z_0 \in \Omega$, choose a zig-zag path $\gamma : [0,1] \to \Omega$ such that $\gamma(0) = \alpha$, $\gamma(1) = z_0$.

just to fix ideas : $[a,b] \rightsquigarrow [0,1]$

(Note: since Ω is connected, we saw last time that any two points can be joined by a zig-zag path – see (13.5) page 8 of Lecture 13.)

(ii) Define
$$\boxed{F(z_0) = \int_{\gamma} f(z)\, dz}$$

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* based on an argument due to Giacinto Morera (1856-1909, Novara, Italy)

Note: This idea is almost verbatim to the one due to Newton and Leibniz (compare with Lecture 11 - page 3).

The only difference is that, over $\mathbb{R}$ there is only one way to go from $\underset{\text{(fixed)}}{a \in \mathbb{R}}$ to $\underset{\text{(varies)}}{t \in \mathbb{R}}$, and over $\mathbb{C}$ there are <u>many ways</u>.

→ The hypothesis from (3) of our theorem ensures that the definition given above $\boxed{F(z_0) = \int_\gamma f(z)\,dz}$ does not depend on which zig-zag path γ we chose.

Now we will prove that $F : \Omega \to \mathbb{C}$, thus defined, is $\mathbb{C}$-differentiable, and $F'(z) = f(z)$. By Theorem (6.1) - page 2 of Lecture 6 (Cauchy-Riemann) - it is sufficient to show that:

[if we write $F(z) = U(x,y) + V(x,y)i$; $x = \mathrm{Re}(z)$, $y = \mathrm{Im}(z)$.]
$\qquad\qquad\quad f(z) = u(x,y) + v(x,y)i$

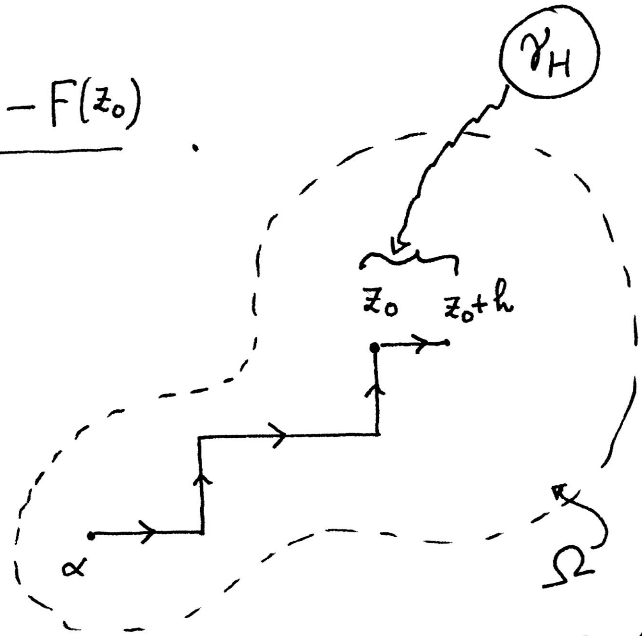
(a) $F_x = U_x + V_x i = f$

(b) $-iF_y = V_y - U_y i = f$

[Note: once we prove (a) and (b), we will automatically obtain the continuity of U_x, U_y, V_x, V_y, since f is assumed to be continuous - hypotheses (i), (ii), (iii) of Theorem (6.1) will be satisfied.]

Verification of (a): $\boxed{F_x = f}$. Let $z_0 \in \Omega$.

Then $$F_x(z_0) = \lim_{\substack{h \to 0 \\ (h \in \mathbb{R})}} \frac{F(z_0+h) - F(z_0)}{h}.$$



$$F(z_0) = \int_\gamma f(z)\,dz \quad \text{and}$$

$$F(z_0+h) = \int_\gamma f(z)\,dz + \int_{\gamma_H} f(z)\,dz$$

[$|h|$ small enough, so: $z_0 + h \in \Omega$;
γ : zig-zag path from α to z_0
$\tilde{\gamma}$: concatenate horizontal line $z_0 - z_0 + h$ to γ.]
$(h \in \mathbb{R})$

$$\boxed{\begin{aligned} \gamma_H &= \text{horizontal line joining } z_0 \text{ and } z_0+h, \text{ e.g.,} \\ \gamma_H(t) &= z_0 + th\ ;\ 0 \le t \le 1 \end{aligned}}$$

Thus, $$\lim_{\substack{h \to 0 \\ (h \in \mathbb{R})}} \frac{F(z_0+h) - F(z_0)}{h} = \lim_{\substack{h \to 0 \\ (h \in \mathbb{R})}} \frac{1}{h} \cdot \int_{\gamma_H} f(z)\,dz$$

$$= \lim_{h \to 0} \frac{1}{h} \int_0^1 \underbrace{f(z_0+th)}_{f(\gamma_H(t))} \cdot \underbrace{h}_{\gamma_H'(t)} \cdot dt = \lim_{h \to 0} \int_0^1 f(z_0+th)\,dt$$

<u>Claim</u>: $$\boxed{\lim_{h \to 0} \int_0^1 f(z_0+th)\,dt = f(z_0)}$$ (Hence, (a) follows)

<u>Proof of the claim.</u> By definition of $\boxed{\lim_{h\to 0}}$, we need to show that: <u>Given $\varepsilon > 0$</u>, we can find $\delta > 0$, so that

$$0 < |h| < \delta \implies \left| \int_0^1 f(z_0 + th)\, dt - f(z_0) \right| < \varepsilon.$$

Since f is continuous (at z_0), there certainly is some $\delta > 0$ so that

$$0 < |k| < \delta \implies |f(z_0 + k) - f(z_0)| < \varepsilon. \qquad (k = th).$$

This δ works for us! as shown below:

If $0 < |h| < \delta$, then $\underset{(0<t<1)}{|th|} < \delta$ and we get:

$$\left| \int_0^1 f(z_0 + th)\, dt - f(z_0) \right| = \left| \int_0^1 \left(f(z_0+th) - f(z_0)\right) dt \right|$$

$$\leqslant \int_0^1 |f(z_0+th) - f(z_0)|\, dt < \int_0^1 \varepsilon \cdot dt = \varepsilon.$$

$\square$

Verification of (b) is exactly the same, just replace $\underset{\text{(horizontal)}}{\gamma_H}$ by $\underset{\text{(vertical)}}{\gamma_V}$.

The theorem is proved.

$\square$

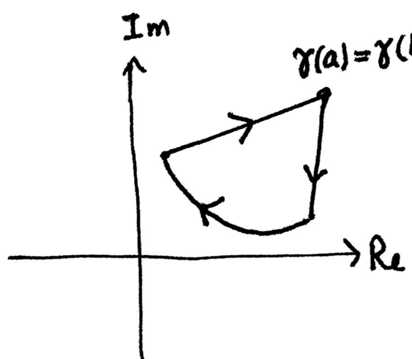
(14.2) Closed paths, simple paths, contours.

Let $\gamma : [a, b] \longrightarrow \mathbb{C}$ be a piecewise smooth path.

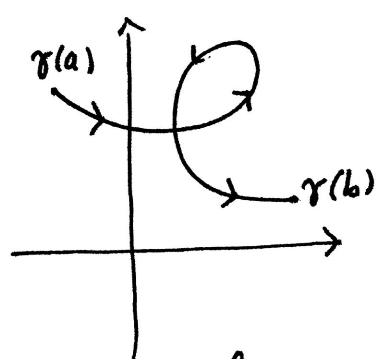
We say γ is closed if $\gamma(a) = \gamma(b)$. We say γ is simple if it does not cross itself, except possibly at end points. That is,

for every t_1, t_2 ; $a < t_1 < t_2 < b$; $\gamma(t_1) \neq \gamma(t_2)$

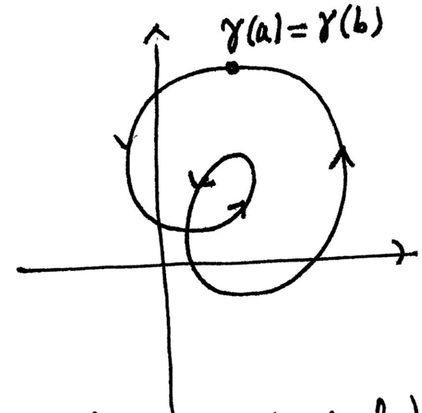
for every t ; $a < t < b$; $\gamma(t) \neq \gamma(a)$ and $\gamma(t) \neq \gamma(b)$.



(a simple closed path)



(Neither simple nor closed)



(closed, not simple)

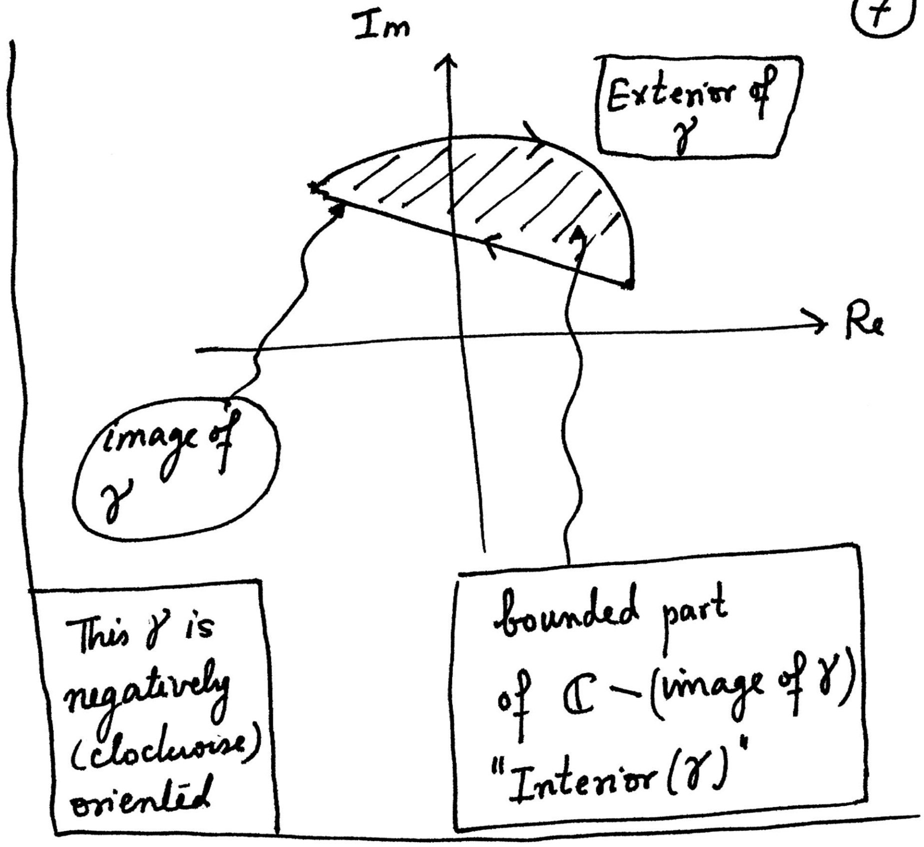
Contour = simple and closed.

(14.3) Interior, exterior and orientation.

Let $\gamma : [a, b] \to \mathbb{C}$ be a contour (i.e., γ is simple and closed). Then $\mathbb{C} \setminus$ image of γ $\left(= \left\{ z \in \mathbb{C} \;\middle|\; \begin{array}{l} z \neq \gamma(t) \\ \text{for any } a \leq t \leq b \end{array} \right\}\right)$

has two connected components – a bounded one and unbounded one.

The bounded component of $\mathbb{C} \setminus (\text{image of } \gamma)$ is called the <u>interior of γ</u>.



The orientation of γ is said to be <u>positive</u> (or <u>counterclockwise</u>) if the <u>interior of γ</u> <u>is to the left</u>, while traversing along γ.

(14.4) Restating our theorem (14.0)-above.

- $\Omega \subseteq \mathbb{C}$ open, connected
- $f : \Omega \to \mathbb{C}$ continuous.

TFAE (acronym for "The following are equivalent")

(1) f admits an antiderivative (on Ω).

(2) For every contour $\gamma : [a,b] \to \Omega$, $\int_\gamma f(z)\,dz = 0$.

(3) For every <u>rectangular contour</u> $\gamma : [a,b] \to \Omega$,

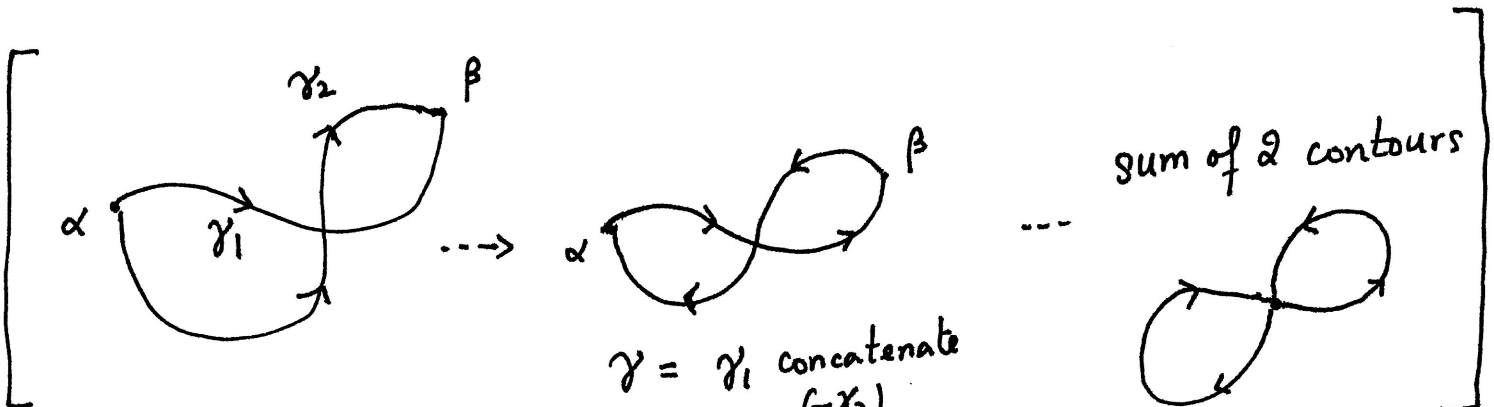
$$\int_\gamma f(z)\,dz = 0.$$

Reason why this is merely a restatement:

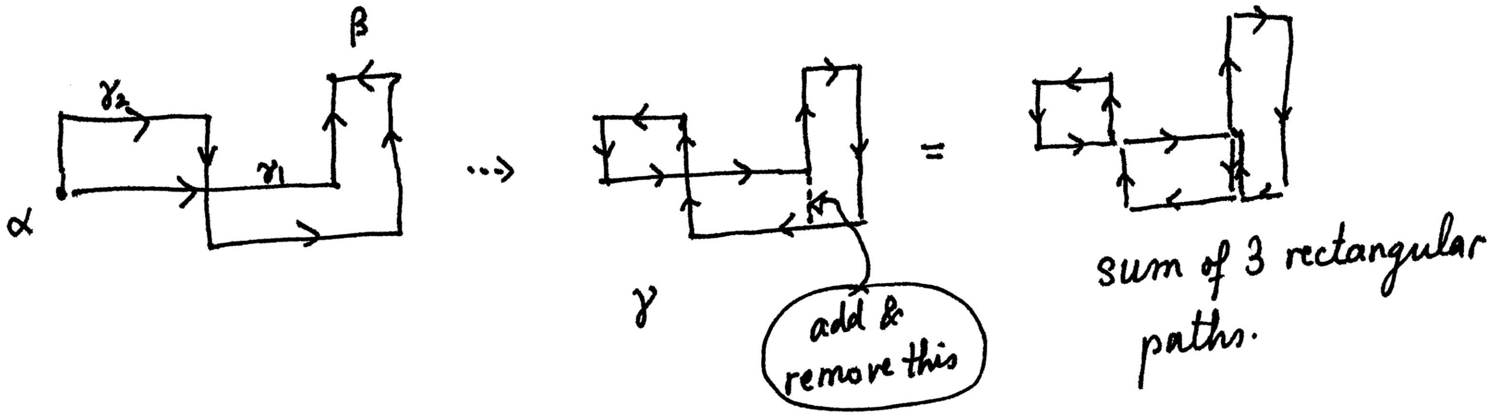
- given γ_1, γ_2 two paths in Ω with same endpoints; $\{\gamma_1$ concatenated with $(-\gamma_2)\} = \gamma$ is a closed path.

$$\int_{\gamma} f(z)\,dz = \int_{\gamma_1} f(z)\,dz + \int_{(-\gamma_2)} f(z)\,dz = \int_{\gamma_1} f(z)\,dz - \int_{\gamma_2} f(z)\,dz.$$

(even though γ may not be simple, we can break it into simple pieces.)



- Similarly if γ_1 and γ_2 are two zig-zag paths joining α & β; $\gamma_1(-\gamma_2) = \gamma$ can be written as sum of rectangular paths.



(14.5) Next time we will prove <u>Cauchy's theorem</u>:

- $\Omega \subseteq \mathbb{C}$ open, connected.
- $f : \Omega \to \mathbb{C}$: <u>$\mathbb{C}$-differentiable</u>.
- γ = a contour in Ω such that <u>$\text{Interior}(\gamma) \subset \Omega$</u>.

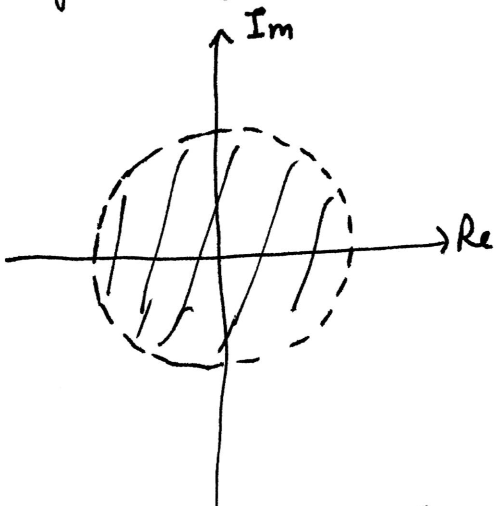
Then

$$\int_\gamma f(z)\, dz = 0.$$

This theorem signifies the importance of those subsets of $\mathbb{C}$ (Ω's) for which the last hypothesis is always true. ($\text{interior}(\gamma) \subset \Omega$).

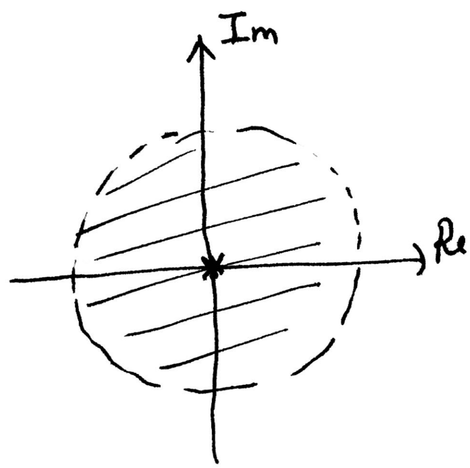
These are called <u>simply-connected</u>:

(14.6) A subset $\Omega \subseteq \mathbb{C}$ is called <u>simply-connected</u> if <u>for every contour</u> $\gamma : [a,b] \to \Omega$, we have: <u>$\text{Interior}(\gamma) \subset \Omega$</u>.



$\Omega = D(0,R) = \{|z| < R\}$

is simply-connected



$\overset{\times}{D}(0,R) = \{0 < |z| < R\}$

is <u>not</u> simply connected.