Lecture 15

(15.0) Recall: last time we introduced the notions of <u>closed path (or loop)</u>, <u>simple path</u>, and a <u>contour</u>. We showed that the following three statements are equivalent, for a continuous function $f : \Omega \to \mathbb{C}$, defined on an open and connected set Ω:

(1) There exists a $\mathbb{C}$-differentiable $F : \Omega \to \mathbb{C}$ such that

$$F'(z) = f(z), \quad \forall z \in \Omega.$$

(2) $$\int_\gamma f(z)\,dz = 0, \text{ for every contour } \gamma \text{ in } \Omega. \text{ (or, even for any closed path } \gamma).$$

(3) $$\int_\gamma f(z)\,dz = 0, \text{ for every rectangular contour } \gamma \text{ in } \Omega.$$

(15.1) Definition. A subset $\Omega \subseteq \mathbb{C}$ is called <u>simply-connected</u> if
(see (14.5) & (14.6)) for every contour γ in Ω, we have:

$$\boxed{\text{Interior}(\gamma) \subset \Omega}$$

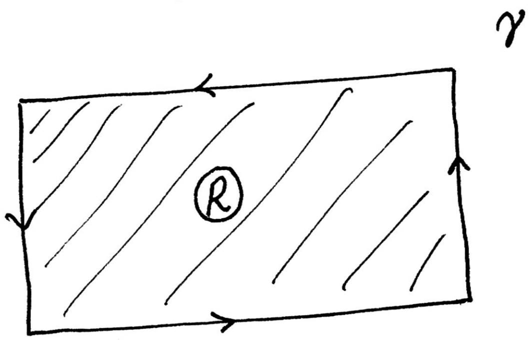
Today we are going to prove the famous Cauchy's Theorem.

(15.2) Cauchy's Theorem.- Let $f : \Omega \to \mathbb{C}$ be a $\mathbb{C}$-differentiable function, where $\Omega \subseteq \mathbb{C}$ is open, connected and simply-connected. Then for every closed path $\gamma : [a, b] \to \Omega$:

$$\boxed{\int_\gamma f(z)\,dz = 0}$$

The idea of the proof below is to first prove the statement assuming γ is a rectangular contour. Then use the theorem from last lecture (14.0) – see equivalent formulation in (14.4), or the previous page – to conclude that $f(z)$ must have an antiderivative $F(z)$. This implies $\int_\gamma f(z)\,dz = F(\gamma(b)) - F(\gamma(a))$, hence equal to 0 if $\gamma(a) = \gamma(b)$.

(15.3)* So, let us assume that γ is a rectangular contour.



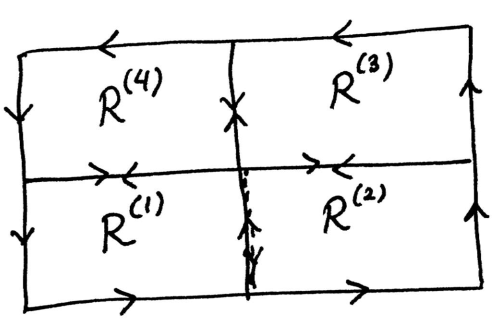
$R = \text{Interior}(\gamma) \subset \Omega$ since Ω is simply-connected.

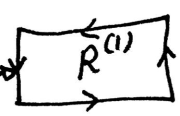
We are going to show that, given any $\varepsilon > 0$;

$$\left| \int_\gamma f(z)\,dz \right| < \varepsilon \cdot (\text{perimeter of } R) \cdot (\text{diagonal of } R).$$

Thus, by necessity, $\int_\gamma f(z)\,dz$ must be 0.

(1) Divide R into 4 equal pieces: $R^{(1)}, R^{(2)}, R^{(3)}, R^{(4)}$. Let $\gamma^{(1)}, \gamma^{(2)}, \gamma^{(3)}, \gamma^{(4)}$ be their respective boundry contours – see picture:



($\gamma^{(1)}$ not  for illustration)

*Optional

Then:
$$\int_{\gamma} f(z)\,dz = \int_{\gamma^{(1)}} f(z)\,dz + \int_{\gamma^{(2)}} f(z)\,dz + \int_{\gamma^{(3)}} f(z)\,dz + \int_{\gamma^{(4)}} f(z)\,dz.$$

(2) Let γ_1 be the one from $\{\gamma^{(1)}, \gamma^{(2)}, \gamma^{(3)}, \gamma^{(4)}\}$ for which $\left|\int_{\gamma^{(j)}} f(z)\,dz\right|$ is the maximum, and R_1 be the rectangle bound by γ_1. Thus:

by triangle inequality

$$\left|\int_{\gamma} f(z)\,dz\right| \leq \sum_{j=1}^{4} \left|\int_{\gamma^{(j)}} f(z)\,dz\right| \leq 4\left|\int_{\gamma_1} f(z)\,dz\right|$$

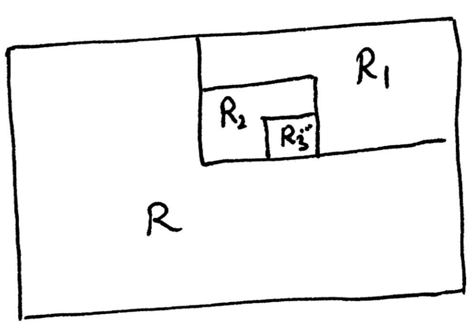
$$\text{diameter}(R) = 2\ \text{diameter}(R_1)$$
$$\text{perimeter}(R) = 2\ \text{perimeter}(R_1)$$

→ Repeat Steps (1) and (2) above with R_1 in the place of R to get R_2; R_2 in the place of R to get R_3, and so on...

We obtain the following inequalities:

$$\boxed{\begin{gathered} \left|\int_{\gamma} f(z)\,dz\right| \leq 4^n \left|\int_{\gamma_n} f(z)\,dz\right| \\ \text{diameter}(R) = 2^n \cdot \text{diameter}(R_n) \\ \text{perimeter}(R) = 2^n \cdot \text{perimeter}(R_n) \\ (n = 1, 2, 3, \ldots) \end{gathered}}$$

A



With these shrinking rectangles at hand, let

$$\{w\} = \bigcap_{n=1}^{\infty} R_n \quad \left(\text{i.e. } w \in R \text{ is the only point that lies in all the rectangles } R \supset R_1 \supset R_2 \supset \cdots\right)$$

[This property is sometimes called- descending chain condition for closed and bounded sets. It is <u>almost axiomatic</u> for the real line: given a nested chain of closed (finite) intervals, $I_0 \supset I_1 \supset \cdots$ such that $\lim_{n\to\infty} \text{length}(I_n) = 0$, there is a unique $x \in \mathbb{R}$ such that $x \in I_n$ for every $n = 0, 1, 2, \ldots$ I will write this in Optional Reading A, for those of you who are interested.]

Now, let there be given some $\varepsilon > 0$.

By definition of $\boxed{\lim_{h\to 0} \frac{f(w+h) - f(w)}{h} = f'(w)}$ we can find $\delta > 0$, such that

$$0 < |z - w| < \delta \implies \left| \frac{f(z) - f(w) - (z-w) f'(w)}{z - w} \right| < \varepsilon.$$

Also take $n > 0$ large enough so that

$$R_n \subset D(w; \delta) \quad (= \{z \mid |z-w| < \delta\}.)$$

Then we can conclude:

$$\left| \int_\gamma f(z)\,dz \right| \leq 4^n \cdot \left| \int_{\gamma_n} f(z)\,dz \right| \quad \text{(see (A) on page 3)}$$

$$= 4^n \left| \int_{\gamma_n} \left(f(z) - f(w) - (z-w) f'(w) \right) dz \right|$$

[This is because, $f(w) + (z-w) f'(w)$ has an antiderivative:
(remember: w is fixed.) $\quad z f(w) + \left(\frac{z^2}{2} - wz\right) f'(w)$; so

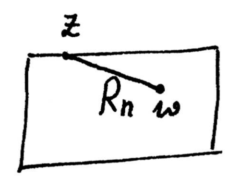
$$\int_{\gamma_n} \left(f(w) + (z-w) f'(w) \right) dz = 0.]$$

$$\leq 4^n \cdot \left(\begin{array}{l} \text{largest value } \left| f(z) - f(w) - (z-w) f'(w) \right| \\ \text{can take; as } z \text{ runs over } \gamma_n \end{array} \right) \cdot \text{length}(\gamma_n)$$

[Important inequality from Lecture 12 – (12.7) - page 9 .]

- Since for $z \in R_n$, $|z-w| < \delta$, we have

$$\left| f(z) - f(w) - (z-w) f'(w) \right| < \varepsilon \cdot |z-w|$$
$$< \varepsilon \cdot \text{diameter}(R_n).$$



$\left(|z-w| < \text{diameter}(R_n) \right)$

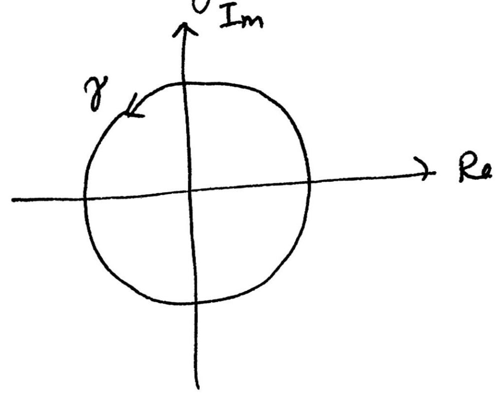
We get $$\left| \int_\gamma f(z)\,dz \right| < 4^n \cdot \varepsilon \cdot \text{diameter}(R_n) \cdot \text{perimeter}(R_n)$$
$$= \varepsilon \cdot \text{diameter}(R) \cdot \text{perimeter}(R) \quad \left(\text{see (A) on page 3}\right)$$

This is exactly what we wanted to show. (line 4 of (15.3) beginning on page 2.)

$\square$

(15.4) Some examples following from Cauchy's theorem.

- $\gamma(t) = e^{it} \quad (0 \leq t \leq 2\pi)$.



$$\int_\gamma e^{z^2}\, dz = 0,$$

$$\int_\gamma \cos(z^3 - 3z)\, dz = 0,$$

$$\int_\gamma \sin(e^z)\, dz = 0.$$

(It would be difficult to prove/compute their antiderivatives!)