Lecture 16

(16.0) Recall - we proved Cauchy's theorem (Lecture 15, Section (15.2)):

$$\boxed{\int_\gamma f(z)\,dz = 0}$$

- γ is a simple closed curve (i.e. contour).
- $f : \Omega \to \mathbb{C}$ is $\mathbb{C}$-differentiable on an open set, which contains γ <u>and</u> interior(γ).

In this lecture, we are going to derive some important consequences of this theorem (all due to Cauchy):

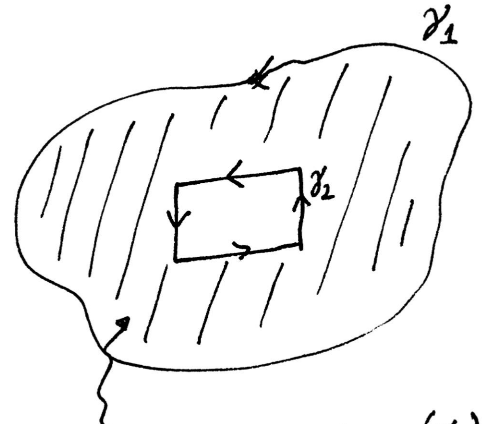
- principle of deformations of contours.
- a stronger version of Cauchy's theorem.
- Cauchy's integral formula.

(16.1) Principle of deformation of contours.

Let $f : \Omega \to \mathbb{C}$ be a $\mathbb{C}$-differentiable function, defined on an open set. Let $\gamma_1, \gamma_2 : [0,1] \to \Omega$ be two positively oriented contours, such that:

$$\begin{cases} \text{Interior}(\gamma_2) \subset \text{Interior}(\gamma_1) \\ \text{Exterior}(\gamma_2) \cap \text{Interior}(\gamma_1) \subset \Omega \end{cases}$$

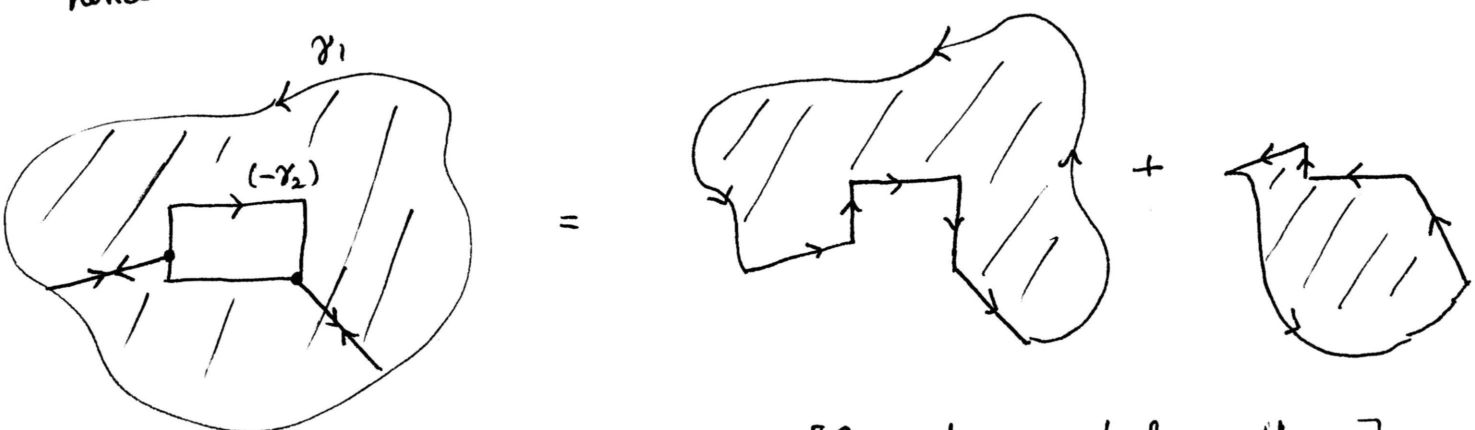
(see picture on the right)



Exterior$(\gamma_2) \cap$ Interior(γ_1) is still within the domain of f, Ω.

Then $$\int_{\gamma_1} f(z)\,dz = \int_{\gamma_2} f(z)\,dz .$$

Proof. $\int_{\gamma_1} f(z)\,dz - \int_{\gamma_2} f(z)\,dz$ can be written as sum of two integrals over contours satisfying the hypotheses of Cauchy's theorem, hence has to be zero. $\square$

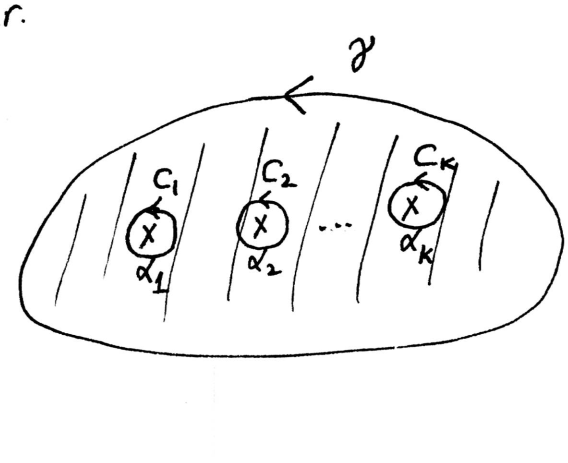


[pick 2 points on γ_1 and "closest" points to these on γ_2 - add & subtract integrals over indicated lines]

[2 contours satisfying the hypotheses of Cauchy's theorem].

Remark. - In practice, this principle allows us to pick the most convenient contour over which we can compute the integral easily, without changing the final answer.

An illustration:



- γ is a positively oriented contour
- $\text{Interior}(\gamma) \setminus \{\alpha_1, \dots, \alpha_k\} \subset \Omega$

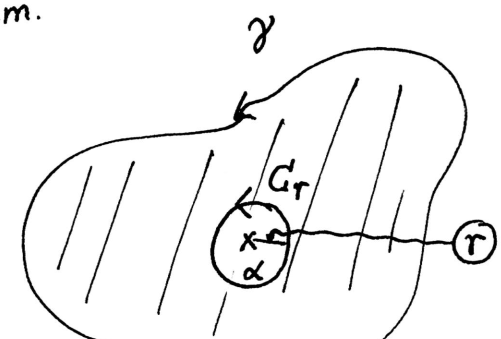
($f : \Omega \to \mathbb{C}$ is $\mathbb{C}$-differentiable)

Thus, f is defined (& is $\mathbb{C}$-differentiable) on Interior(γ), except possibly at a finite set of points $\{\alpha_1, \dots, \alpha_k\} \subset$ Interior(γ).

Let $C_1, \dots, C_k$ be (counterclockwise) circles centered at $\alpha_1, \dots, \alpha_k$ respectively of small enough radii. (see picture on the previous page).

Then:
$$\int_\gamma f(z)\,dz = \sum_{j=1}^{k} \int_{C_j} f(z)\,dz .$$

(16.2) A stronger version of Cauchy's theorem.



- $f : \Omega \to \mathbb{C}$; $\mathbb{C}$-differentiable on an open set Ω.
- $\gamma : [0,1] \to \Omega$; (positively oriented) contour
- $\alpha \in$ Interior(γ) s.t. Interior$(\gamma) - \{\alpha\} \subset \Omega$

[f is defined on Interior(γ) except possibly at α.]

If $\lim_{z\to\alpha} (z-\alpha) f(z) = 0$, then: $\int_\gamma f(z)\,dz = 0$.

Proof.- $\int_\gamma f(z)\,dz = \int_{C_r} f(z)\,dz$; $C_r = \alpha + r.e^{i\theta}$ $(0 \le \theta \le 2\pi)$ is the counterclockwise circle of radius $r \in \mathbb{R}_{>0}$, centered at α.

(by contour deformation principle - (16.1))

Since $\lim\limits_{z \to \alpha} (z-\alpha) f(z) = 0$, given any $\varepsilon > 0$, we can find $\delta > 0$ so that

$$0 < |z-\alpha| \le \delta \Rightarrow |(z-\alpha) f(z)| < \varepsilon.$$

Hence

$$\left| \int_{C_\delta} f(z)\, dz \right| \le \overset{\text{bound for } f \text{ along } C_\delta}{\frac{\varepsilon}{\cancel{\delta}}} \cdot \overset{\text{length of } C_\delta}{\underbrace{2\pi\cancel{\delta}}_{\text{length of } C_\delta}}$$

(by inequality (12.9) - page 9 of Lecture 12.)

(radius r is replaced by δ)

(for z on the circle C_δ, $|z-\alpha| = \delta$.)

$$\left|\int_0^{2\pi} f(\alpha+re^{i\theta})\, r i e^{i\theta}\, d\theta\right| \le \int_0^{2\pi} |f(\underbrace{\alpha+re^{i\theta}}_{=z})| \underbrace{r}_{|z-\alpha|<\delta}\, d\theta \le \varepsilon 2\pi$$

$$= 2\pi\varepsilon.$$

We have, thus, shown that $\left| \int_\gamma f(z)\, dz \right| \le 2\pi\varepsilon$ for every $\varepsilon \in \mathbb{R}_{>0}$.

Therefore, it must be zero. $\square$

(16.3) Cauchy's integral formula.

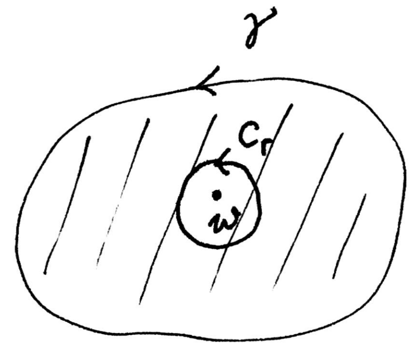
[compare with Example II of Section (13.1) - Lecture 13, page 2.]

- $f : \Omega \to \mathbb{C}$, a $\mathbb{C}$-differentiable function defined on an open set Ω.
- $\gamma : [a, b] \to \Omega$, a positively oriented contour such that Interior$(\gamma) \subset \Omega$.
- $w \in$ Interior(γ).

Then

$$\boxed{\int_\gamma \frac{f(z)}{z-w}\, dz = 2\pi i\, f(w)}$$

Proof.



Let $g(z) = \dfrac{f(z) - f(w)}{z - w}$.

$g(z)$ is then a $\mathbb{C}$-differentiable function, defined on Interior $(\gamma) \setminus \{w\}$; and γ itself. Moreover, by continuity of f: $\lim\limits_{z \to w} (z - w)\, g(z) = \lim\limits_{z \to w} (f(z) - f(w)) = 0$.

Hence, the hypotheses for the strong form of Cauchy's theorem, (16.2) above, are met – for $g(z)$, γ and $\alpha = w$ there. We conclude:

$$0 = \int_\gamma g(z)\, dz = \int_\gamma \left(\frac{f(z)}{z-w} - \frac{f(w)}{z-w} \right) dz$$

$$\Rightarrow \int_\gamma \frac{f(z)}{z-w}\, dz = f(w) \cdot \int_\gamma \frac{1}{z-w}\, dz .$$

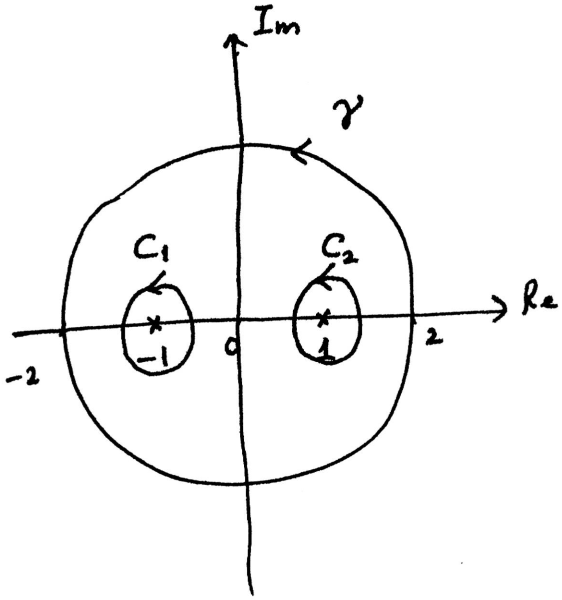
<u>Computation of $\displaystyle\int_\gamma \frac{1}{z-w}\, dz$:</u> By the principle of deformation of contours, we can replace γ by $C_r : w + e^{i\theta}$ $(0 \leq \theta \leq 2\pi)$.

$$\int_\gamma \frac{1}{z-w}\, dz = \int_{C_r} \frac{1}{z-w}\, dz = \int_0^{2\pi} \frac{1}{e^{i\theta}} \cdot i \cdot e^{i\theta}\, d\theta = 2\pi i .$$

$$\Rightarrow \int_\gamma \frac{f(z)}{z-w}\, dz = 2\pi i\, f(w) . \qquad \square$$

(16.4) Example. $f(z) = \dfrac{e^z}{z^2-1}$. $\Omega = \mathbb{C} \setminus \{\pm 1\}$

$\gamma(\theta) = 2 \cdot e^{i\theta}$; $0 \leq \theta \leq 2\pi$.



$$\int_\gamma \frac{e^z}{z^2-1}\, dz$$

contour deformation

$$= \int_{C_1} \left(\frac{e^z}{z-1}\right) \frac{1}{z+1}\, dz + \int_{C_2} \left(\frac{e^z}{z+1}\right) \frac{1}{z-1}\, dz$$

[C_1 = circle of radius < 1 around -1.
C_2 = " " " " " " $+1$.
both counterclockwise.]

$$= 2\pi i \left(\left[\frac{e^z}{z-1}\right]_{\substack{\text{set} \\ z=-1}} + \left[\frac{e^z}{z+1}\right]_{\substack{\text{set} \\ z=1}} \right)$$

(Cauchy's integral formula)

$$= 2\pi i \left(\frac{e^{-1}}{-2} + \frac{e^{1}}{2} \right) = \frac{e - e^{-1}}{2} \cdot 2\pi i.$$

$$= \pi i \,(e - e^{-1}).$$