Lecture 17

(17.0) Let $\Omega \subseteq \mathbb{C}$ be an open set, $f : \Omega \to \mathbb{C}$ a $\mathbb{C}$-differentiable function; $\gamma : [a,b] \to \Omega$ a positively oriented contour such that $\text{Interior}(\gamma) \subset \Omega$.

Recall that in the last two lectures we proved:

Cauchy's Theorem. $$\boxed{\int_\gamma f(z)\, dz = 0.}$$

Cauchy's integral formula: for every $w \in \text{Interior}(\gamma)$:

$$\boxed{\int_\gamma \frac{f(z)}{z-w}\, dz = 2\pi i\, f(w)}$$

(17.1) Today we are going to prove a generalization of Cauchy's integral formula. With $\boxed{\Omega, f, \gamma, w}$ as above:

For every $n \in \mathbb{Z}_{\geq 0}$:

$$\boxed{\frac{f^{(n)}(w)}{n!} = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{(z-w)^{n+1}}\, dz}$$

(1) For $n=0$ this is the integral formula that we proved in the previous lecture.

(2) This generalization implies, in particular, that <u>all the higher derivatives $f^{(2)}(w), f^{(3)}(w), \ldots$ exist</u>. That is:

$$\boxed{f \text{ is } \mathbb{C}\text{-differentiable} \implies f \text{ is } \mathbb{C}\text{-differentiable to all orders}}$$

Hence, (after we prove the formula $\boxed{\dfrac{f^{(n)}(w)}{n!} = \dfrac{1}{2\pi i}\displaystyle\int_\gamma \dfrac{f(z)}{(z-w)^{n+1}}\,dz}$)

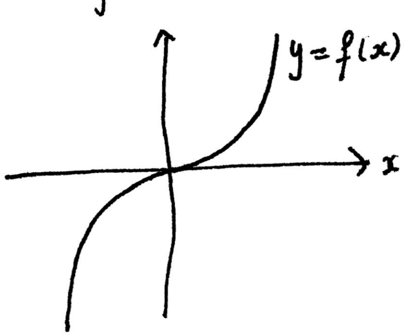
we will use the term <u>holomorphic</u> instead of (or, besides) <u>$\mathbb{C}$-differentiable</u>.

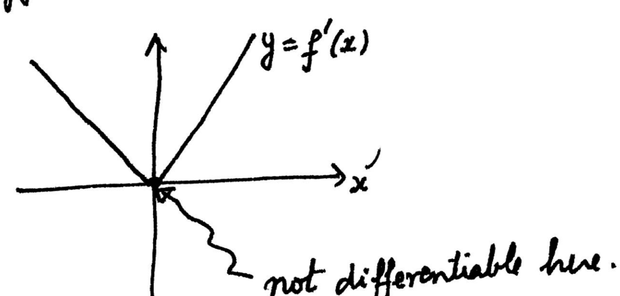
(3) "Once differentiable, always differentiable" — is <u>false over $\mathbb{R}$</u>.

e.g. $\underset{(x \in \mathbb{R})}{f(x)} = \begin{cases} -x^2 & ; x<0 \\ x^2 & ; x \geq 0 \end{cases}$ is once differentiable, with

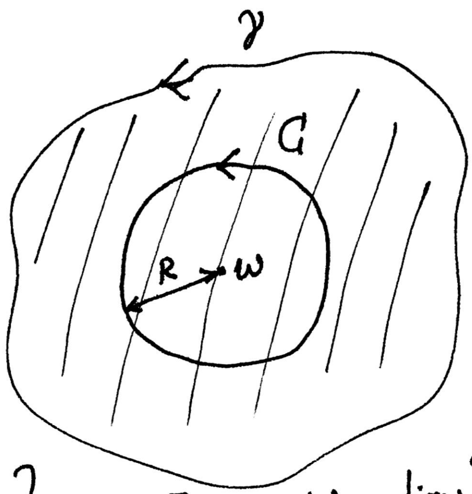
continuous $f'(x) = \begin{cases} -2x \,; & x<0 \\ 2x \,; & x \geq 0 \end{cases}$; i.e. $f'(x) = 2 \cdot |x|$.

But $f'(x)$ is no longer differentiable at $x=0$.





(17.2)* Proof of (17.1) formula



- Pick $\boxed{R>0}$ so that $\underset{\{z \mid |z-w|<R\}}{D(w;R)} \subset \text{Interior}(\gamma)$

Let $C(\theta) = w + Re^{i\theta}$ $(0 \leq \theta \leq 2\pi)$

- Let $M = \max\left\{ |f(z)| : z = C(\theta) \; (0 \leq \theta \leq 2\pi) \right\}$

[Contour deformation (16.1): replace γ by C.]

[remember: a continuous, real-valued function on a closed & bounded interval attains its max/min values there. I will write a proof of this in Optional Reading A.]

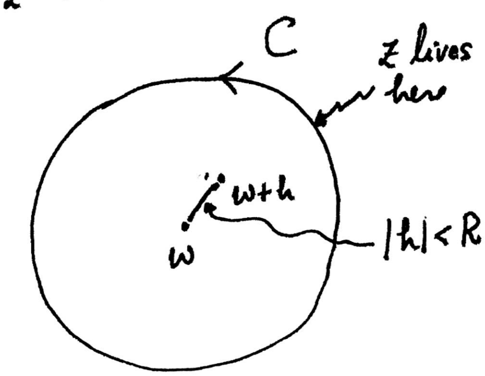
To prove:
$$\boxed{f^{(n)}(w) = \frac{n!}{2\pi i} \int_C \frac{f(z)}{(z-w)^{n+1}}\, dz} \quad \forall\, n = 0, 1, 2, \ldots$$

[Proof will be by induction on n.]

$n=0$ case proved last time - see (16.3) - page 4 of lecture 16.

$n=1$ Case: (always take $h \in \mathbb{C}$ so that $|h|<R$):

$$\frac{f(w+h) - f(w)}{h} \overset{\text{by } n=0 \text{ case}}{=} \frac{1}{h}\frac{1}{2\pi i} \int_C \left(\frac{f(z)}{z-(w+h)} - \frac{f(z)}{z-w} \right) dz$$



$$= \frac{1}{2\pi i} \int_C f(z) \cdot \frac{(\cancel{(z-w)} - \cancel{(z-w}-h))}{h(z-w-h)(z-w)}\, dz$$

$$= \frac{1}{2\pi i} \int_C \frac{f(z)}{(z-w-h)(z-w)}\, dz$$

* Optional

Thus $$\frac{f(w+h)-f(w)}{h} - \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-w)^2}\,dz$$

$$= \frac{1}{2\pi i}\int_C f(z)\cdot \underbrace{\left(\frac{1}{(z-w-h)(z-w)} - \frac{1}{(z-w)^2}\right)}_{\frac{h}{(z-w)^2(z-w-h)}} dz$$

Bound on the integrand: (use $|z-w-h| \geq |z-w| - |h| = R - |h|$):

$$\left|\frac{f(z)\cdot h}{(z-w)^2(z-w-h)}\right| \leq \frac{M\cdot|h|}{R^2(R-|h|)}.$$

$$\Rightarrow \left|\frac{f(w+h)-f(w)}{h} - \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-w)^2}\,dz\right| \leq \frac{1}{2\pi}\,\frac{M\cdot|h|}{R^2(R-|h|)}\cdot \underbrace{2\pi R}_{\text{length}(C)}$$

$$= \frac{M\cdot|h|}{R\cdot(R-|h|)} \to 0 \text{ as } h\to 0.$$ (by inequality (12.9) – page 9 of Lecture 12)

Hence, $$\underbrace{\lim_{h\to 0}\frac{f(w+h)-f(w)}{h}}_{f'(w)} = \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-w)^2}\,dz.$$

[General case is exactly the same – and the computation given below also implies the existence of the $\lim_{h\to 0}$.]

General case : [induction step]

Assume $f^{(n)}(w) = \dfrac{n!}{2\pi i}\displaystyle\int_C \dfrac{f(z)}{(z-w)^{n+1}}\,dz$,

and <u>prove</u> $f^{(n+1)}(w) = \dfrac{(n+1)!}{2\pi i}\displaystyle\int_C \dfrac{f(z)}{(z-w)^{n+2}}\,dz$.

[The calculation is exactly as on the previous page. You will have to use the binomial formula:

$$(z-w-h)^{n+1} = \sum_{j=0}^{n+1} \binom{n+1}{j} (-1)^j h^j (z-w)^{n+1-j}$$ twice.

I am only going to write the final outcome of a calculation — leaving the task of filling-in detailed steps - to you.*]

- $$\frac{f^{(n)}(w+h) - f^{(n)}(w)}{h} = \frac{n!}{2\pi i}\int_C \left(\sum_{k=1}^{n+1} (-1)^{k-1} h^{k-1} (z-w)^{n+1-k} \cdot \binom{n+1}{k}\right) \cdot \frac{f(z)\,dz}{(z-w-h)^{n+1}(z-w)^{n+1}}$$

- $$\frac{f^{(n)}(w+h) - f^{(n)}(w)}{h} - \frac{(n+1)!}{2\pi i}\int_C \frac{f(z)}{(z-w)^{n+2}}\,dz =$$

$$= \frac{n!}{2\pi i}\int_C \frac{f(z)}{(z-w)^{n+2}(z-w-h)^{n+1}} \cdot \left\{ \sum_{l=1}^{n+1} (-1)^{l-1} h^{l} (z-w)^{n+1-l} \cdot \left((n+1)\binom{n+1}{l} - \binom{n+1}{l+1}\right)\right\} dz$$

(convention - when $l = n+1$: $\binom{n+1}{n+2} = 0$.)

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* <u>if</u> you are interested.

$$\bullet \quad \left| \frac{f^{(n)}(w+h) - f^{(n)}(w)}{h} - \frac{(n+1)!}{2\pi i} \int_C \frac{f(z)}{(z-w)^{n+2}}\, dz \right|$$

$$\leq \frac{n!}{\cancel{2\pi}} \frac{M \cdot |h| \cdot \cancel{2\pi R}}{R^{n+\cancel{2}1} (R-|h|)^{n+1}} \cdot \left\{ \sum_{l=1}^{n+1} |h|^{l-1} R^{n+1-l} \left((n+1)\binom{n+1}{l} - \binom{n+1}{l+1} \right) \right\}$$

$$\longrightarrow 0 \quad \text{as } h \to 0.$$

Hence $\displaystyle \lim_{h \to 0} \frac{f^{(n)}(w+h) - f^{(n)}(w)}{h}$ exists (so $f^{(n)}$ is $\mathbb{C}$-differentiable)

and is equal to $\displaystyle \frac{(n+1)!}{2\pi i} \int_C \frac{f(z)}{(z-w)^{n+2}}\, dz$.