# Lecture 18

(18.0) Recall that we proved Cauchy's theorem, integral formula, and its generalization:

- $\Omega \subseteq \mathbb{C}$ is an open set.
- $f : \Omega \to \mathbb{C}$ is a complex differentiable (also called <u>holomorphic</u>) function.
- $\gamma : [a,b] \longrightarrow \Omega$ is a counterclockwise contour such that interior $(\gamma) \subset \Omega$.

Cauchy's Theorem:
$$\boxed{\int_\gamma f(z)\,dz = 0}$$

Cauchy's integral formula: for any $w \in$ Interior $(\gamma)$:

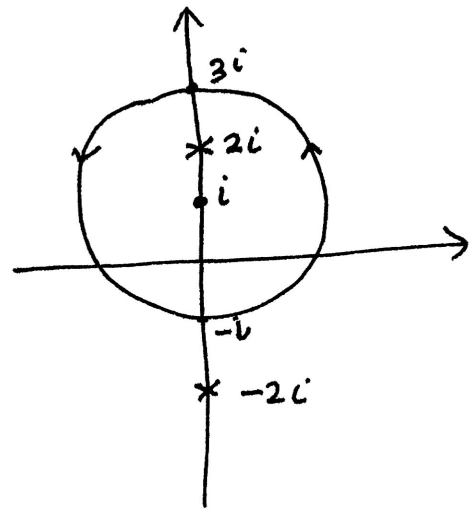
$$\boxed{\int_\gamma \frac{f(z)}{(z-w)^{n+1}}\,dz = \frac{2\pi i}{n!} f^{(n)}(w) \quad \text{for every } n = 0, 1, 2, \ldots}$$

( $f^{(n)}(w) = n^{th}$ order derivative of $f$, evaluated at $w$. )

(18.1) Example. Let $\tilde{f}(z) = \dfrac{1}{(z^2+4)^2}$ and

$\gamma$ = counterclockwise oriented circle of radius 2, centered at $i$.

Thus $\gamma : [0, 2\pi] \to \mathbb{C}$ ; $\gamma(\theta) = i + 2 \cdot e^{i\theta}$




Compute $\displaystyle\int_\gamma \tilde{f}(z)\,dz$.

Sol. $\cancel{f(z)=}\ \dfrac{1}{\left((z+2i)(z-2i)\right)^2} = \tilde{f}(z)$

is not defined at $z = \pm 2i$.

Only one of these points is in the interior of $\gamma$ (see picture above), namely $+2i$.

$$\int_\gamma \frac{1}{(z^2+4)^2}\,dz = \int_\gamma \left(\frac{1}{(z+2i)^2}\right) \cdot \frac{1}{(z-2i)^2}\,dz$$

$$= \frac{2\pi i}{1!} \left.\frac{d}{dz}\left(\frac{1}{(z+2i)^2}\right)\right|_{z=2i}$$

$\left[\text{take } n=1,\ f(z) = \dfrac{1}{(z+2i)^2},\ w = 2i \text{; in the integral formula on the last page}\right]$

$$= 2\pi i \cdot \left[\frac{-2}{(z+2i)^3}\right]_{z=2i}$$

$$= \frac{-4\pi i}{(4i)^3} = \frac{\pi}{16}.$$

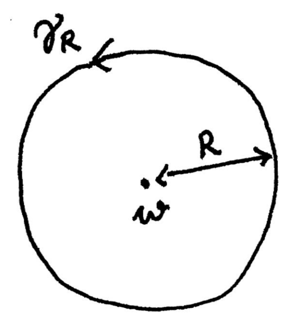
(18.2) Liouville's Theorem*. Let $f(z)$ be a <u>holomorphic function</u>, <u>defined on the entire complex plane</u> (i.e. domain of $f = \mathbb{C}$). Assume that $f(z)$ <u>is bounded</u>: (i.e. there exists $M \in \mathbb{R}_{>0}$ s.t.

$|f(z)| \leq M$ for every $z \in \mathbb{C}$.)

Then: <u>$f(z)$ is constant</u>.

Proof. We are going to prove that $f'(w) = 0$ for every $w \in \mathbb{C}$, which will then imply that $f$ is a constant function.

So, let $w \in \mathbb{C}$. For $R \in \mathbb{R}_{>0}$, let $\gamma_R(\theta) = w + R \cdot e^{i\theta}$, $(0 \leq \theta \leq 2\pi)$ be the counterclockwise circle of radius $R$, centered at $w$.




By Cauchy's integral formula:

$$f'(w) = \frac{1}{2\pi i} \int_{\gamma_R} \frac{f(z)}{(z-w)^2}\, dz.$$

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* Joseph Liouville (1809–1882, Paris). This theorem is actually due to Cauchy (1844). It was called Liouville's Theorem by Borchardt, who learnt it from Liouville's 1847 Lectures in Paris.

Note : Interior $(\gamma_R) = D(w; R) \subset$ domain of $f$, since, by our hypothesis, domain of $f = \mathbb{C}$.

The integral $\displaystyle\int_{\gamma_R} \frac{f(z)}{(z-w)^2}\,dz$ does not depend on $R$ - also by Cauchy's formula (or, the principle of contour deformation) [Lecture 16, section (16.1)].

Thus we get :

$$|f'(w)| = \left| \frac{1}{2\pi i} \int_{\gamma_R} \frac{f(z)}{(z-w)^2}\,dz \right|$$

$$\leq \frac{1}{2\pi} \underbrace{\frac{M}{R^2}}_{\left|\frac{f(z)}{(z-w)^2}\right| \leq \frac{M}{R^2} \text{ for } z \text{ on } \gamma_R} \cdot \underbrace{2\pi R}_{\text{length}(\gamma_R)}$$

$$= \frac{M}{R}.$$

Since we can make $R$ as large as we want, we conclude that

$|f'(w)| \leq \dfrac{M}{R}$ for every $R \in \mathbb{R}_{>0}$ $\Rightarrow$ $f'(w) = 0$. $\square$

[$M \in \mathbb{R}_{>0}$ is fixed - see the statement of the theorem on last page]

(18.3) Fundamental Theorem of Algebra.

Let $n \geq 1$ and let $P(z) = a_n z^n + a_{n-1} z^{n-1} + \dots + a_1 z + a_0$ (here $a_0, a_1, \dots, a_n \in \mathbb{C}$, $a_n \neq 0$) be a polynomial of degree $n$. Then, there exists $\alpha \in \mathbb{C}$ such that $P(\alpha) = 0$.

Proof. The proof of this theorem is by contradiction. Let us assume that $P(w) \neq 0$ (for every $w \in \mathbb{C}$). If this is the case, then $f(z) = \dfrac{1}{P(z)}$ will be defined for every $z \in \mathbb{C}$ (i.e. domain of $f = \mathbb{C}$).

We are going to show that <u>$f(z)$ is also bounded</u>. Once we prove that, the hypotheses of Liouville's Theorem (18.2) will be satisfied, and we will be able to conclude that $f(z) = C$ is a constant function ($C \in \mathbb{C}$ some constant). But this is a contradiction to the assumption that $P(z)$ is a polynomial of degree $\geq 1$.

Proof that $f(z) = \dfrac{1}{P(z)}$ is bounded:

$$|P(z)| = \left| a_n z^n + a_{n-1} z^{n-1} + \dots + a_1 z + a_0 \right|$$

$$= |a_n z^n| \cdot \left| 1 + \frac{a_{n-1}}{a_n z} + \frac{a_{n-2}}{a_n z^2} + \dots + \frac{a_1}{a_n z^{n-1}} + \frac{a_0}{a_n z^n} \right|$$

$$\geq |a_n z^n| \cdot \left| 1 - \left|\frac{a_{n-1}}{a_n z}\right| - \left|\frac{a_{n-2}}{a_n z^2}\right| - \dots - \left|\frac{a_0}{a_n z^n}\right| \right| .$$

(use triangle inequality:
$|\alpha \pm \beta| \geq |\alpha| - |\beta|$)

Let $R \in \mathbb{R}_{>0}$ be large enough so that each term

$$\left|\frac{a_{n-1}}{a_n z}\right|, \left|\frac{a_{n-2}}{a_n z^2}\right|, \dots, \left|\frac{a_0}{a_n z^n}\right| < \frac{1}{2n}$$ ; for every $z$ such that $|z| > R$.

Then we can conclude:

- for $|z| > R$ ; $\quad |P(z)| \geq |a_n| \cdot R^n \cdot \frac{1}{2}$ .

  Thus $|f(z)| = \dfrac{1}{|P(z)|} \leq \dfrac{2}{|a_n| \cdot R^n}$ .

- for $|z| \leq R$ : $\quad |f(z)|$ is a continuous function, on the closed and bounded set $\{z \mid |z| \leq R\}$, hence must attain absolute max there - say $M$.

Putting the two arguments together

For every $z \in \mathbb{C}$ : $\quad |f(z)| \leq \mathrm{MAX}\left(M, \dfrac{2}{|a_n| R^n}\right)$ is bounded. $\square$

(18.4) Corollary. – Let $P(z) = z^n + a_{n-1}z^{n-1} + \dots + a_1 z + a_0$ be a polynomial of degree $n \geq 1$. Here, $a_0, a_1, \dots, a_{n-1} \in \mathbb{C}$. Then there exist $\alpha_1, \alpha_2, \dots, \alpha_n \in \mathbb{C}$ such that

$$P(z) = (z-\alpha_1)(z-\alpha_2)\cdots(z-\alpha_n).$$

[ $\alpha_1, \dots, \alpha_n$ are called roots of the polynomial equation $P(z)=0$. These need not be distinct. ]

Proof. Use Fundamental Theorem of Algebra to conclude that there exists $\alpha \in \mathbb{C}$ such that $P(\alpha)=0$.

Claim : $\quad P(z) = (z-\alpha)\cdot Q(z)$

where $Q(z)$ is a polynomial of degree $n-1$.
[ $Q(z) = z^{n-1} + b_{n-2}z^{n-2} + \dots + b_0$. ]

Assuming this claim, we can invoke induction to write

$$Q(z) = (z-\alpha_2)(z-\alpha_3)\cdots(z-\alpha_n),$$

which proves our corollary.

Proof of the claim. I. Divide $P(z)$ by $(z-\alpha)$ ( Euclid's "long division" algorithm). Let $c \in \mathbb{C}$ be the remainder:

$$P(z) = (z-\alpha)\cdot Q(z) + c.$$

Set $z=\alpha$, $\quad P(\alpha) = 0\cdot Q(\alpha) + c \Rightarrow c = 0.$

$\boxed{P(\alpha)=0}$

Proof of the claim II.

$$P(z) = P(z) - P(\alpha) \qquad (\text{since } P(\alpha) = 0)$$

$$= (z^n - \alpha^n) + a_{n-1}(z^{n-1} - \alpha^{n-1}) + \dots + a_1(z-\alpha) + (a_0 - a_0)$$

$$= (z-\alpha)\underbrace{\left[\frac{z^n-\alpha^n}{z-\alpha} + a_{n-1}\frac{z^{n-1}-\alpha^{n-1}}{z-\alpha} + \dots + a_1\right]}$$

$Q(z)$ - still polynomial - since

$$\frac{z^K - \alpha^K}{z-\alpha} = z^{K-1} + z^{K-2}\alpha + \dots + \alpha^{K-1}$$

□