Lecture 19

(19.0) Recall that last time we used Cauchy's theorem and the integral formula to prove

(1) Liouville's Theorem.- A bounded holomorphic function defined on the entire complex plane has to be constant.

(2) Fundamental Theorem of Algebra.- Every polynomial over complex numbers is a product of linear (degree 1) polynomials.

Definition.- A holomorphic function defined on the entire complex plane is called <u>an entire function</u>.

Examples of entire functions :

- Polynomial functions , e^z , $\sin(z)$, $\cos(z)$ are entire.
- Rational functions are not entire :

$$g(z) = \frac{P(z)}{Q(z)} \quad ; \text{ P and Q are polynomials,}$$

$$\text{Domain of } g = \mathbb{C} - \underbrace{\{z \mid Q(z) = 0\}}_{\substack{\text{finite set - contains at most} \\ \text{degree}(Q) \text{ complex numbers.}}}.$$

(19.1) Partial fraction decomposition of a rational function.

Let $g(z) = \dfrac{P(z)}{Q(z)}$ be a rational function with degree $(P) = n$

degree $(Q) = m$

and assume that $\underline{m > n}$.

(Remark.- if degree $(P) \geq$ degree (Q), we can perform Euclid's "long division" algorithm to write $\dfrac{P}{Q} = \underset{\substack{\uparrow \\ \text{polynomial}}}{A} + \dfrac{B}{Q}$.) $\longleftarrow$ degree$(B) <$ degree(Q)

By Fundamental Theorem of Algebra, we can write $Q(z)$ as a product of linear terms. Since $Q(z)$ may have repeated roots, in general, this means: there are ℓ distinct complex numbers, say $\alpha_1, \alpha_2, \ldots, \alpha_\ell \in \mathbb{C}$ and ℓ positive integers $m_1, m_2, \ldots, m_\ell \in \mathbb{Z}_{\geq 1}$

such that $$Q(z) = c\,(z-\alpha_1)^{m_1} \cdot (z-\alpha_2)^{m_2} \cdot \ldots \cdot (z-\alpha_\ell)^{m_\ell}$$

- $c \in \mathbb{C}$, $c \neq 0$, is the coefficient of z^m in $Q(z)$.
- degree $(Q) = m = m_1 + m_2 + \ldots + m_\ell$.
- m_j is called <u>the multiplicity of the root α_j</u> of $Q(z)$. $(1 \leq j \leq \ell)$.

We are also going to rescale $\underline{c = 1}$. (without loss of generality).

After all this preparation, our rational function takes the following form: $g(z) = \dfrac{P(z)}{(z-\alpha_1)^{m_1} \cdots (z-\alpha_\ell)^{m_\ell}}$; $\boxed{\begin{array}{c}\text{degree}(P) \\ < m \\ (m = m_1 + \dots + m_\ell)\end{array}}$

Partial Fractions decomposition of $g(z)$:

$$\boxed{\begin{aligned} g(z) = \; & \frac{A_{1,1}}{z-\alpha_1} + \frac{A_{1,2}}{(z-\alpha_1)^2} + \dots + \frac{A_{1,m_1}}{(z-\alpha_1)^{m_1}} \\ & + \frac{A_{2,1}}{z-\alpha_2} + \frac{A_{2,2}}{(z-\alpha_2)^2} + \dots + \frac{A_{2,m_2}}{(z-\alpha_2)^{m_2}} \\ & + \dots + \frac{A_{\ell,1}}{z-\alpha_\ell} + \frac{A_{\ell,2}}{(z-\alpha_\ell)^2} + \dots + \frac{A_{\ell,m_\ell}}{(z-\alpha_\ell)^{m_\ell}} \end{aligned}} \quad -(*)$$

(here, $A_{1,1}, \dots, A_{\ell,m_\ell} \in \mathbb{C}$.)

(19.2) Example. Let $g(z) = \dfrac{z^2+2}{(z-1)(z+1)^2}$. [so, $\alpha_1 = 1$, $m_1 = 1$
$\alpha_2 = -1$, $m_2 = 2$
$m = 1 + 2 = 3$
$> \deg(\text{Numerator}) = 2$.]

$$g(z) = \frac{z^2+2}{(z-1)(z+1)^2} = \frac{A}{z-1} + \frac{B}{z+1} + \frac{C}{(z+1)^2}.$$

Problem. – Compute A, B and C.

Solution.- Clear the denominator:

$$z^2+2 = A(z+1)^2 + B(z-1)(z+1) + C(z-1).$$

- Set $z=1$ to get: $\quad 3 = A\cdot 4 \Rightarrow A = \frac{3}{4}$.
- Set $z=-1$ to get: $\quad 3 = C\cdot 2 \Rightarrow C = \frac{3}{2}$.

So, $z^2+2 = \frac{3}{4}(z+1)^2 + B(z^2-1) + \frac{3}{2}(z-1)$. Compare coefficients of z^2:

$$1 = \frac{3}{4} + B \Rightarrow B = \frac{1}{4}.$$

Hence,

$$\frac{z^2+2}{(z-1)(z+1)^2} = \frac{3/4}{z-1} + \frac{1/4}{z+1} + \frac{3/2}{(z+1)^2}.$$

(19.3) Using Cauchy's integral formula to compute partial fractions.

As an easy application of Cauchy's integral formula (or, by a quick and equally easy direct verification), we have

$$\frac{1}{2\pi i}\int_\gamma \frac{1}{(z-\alpha)^k}\,dz = \begin{cases} 1 & \text{if } k=1 \\ 0 & \text{if } k\neq 1 \end{cases}.$$

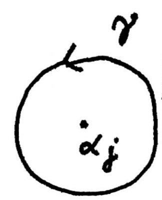
($\gamma(\theta) = \alpha + r\cdot e^{i\theta}$ $(0\leq\theta\leq 2\pi)$

$\gamma =$  of radius $r\in\mathbb{R}_{>0}$)

Thus, in the partial fractions decomposition $(*)$ of page 3 above, if we want to compute, say $A_{j,p}$:

$(1 \leq j \leq \cancel{p}\,\ell \; ; \; 1 \leq p \leq m_j)$

- multiply both sides of $(*)$ by $(z - \alpha_j)^{p-1}$.
- integrate both sides : $\dfrac{1}{2\pi i} \displaystyle\int_\gamma - \, dz$; where γ is a small circle (counterclockwise) centered at α_j (small enough so that no other α_K is in the interior of γ).



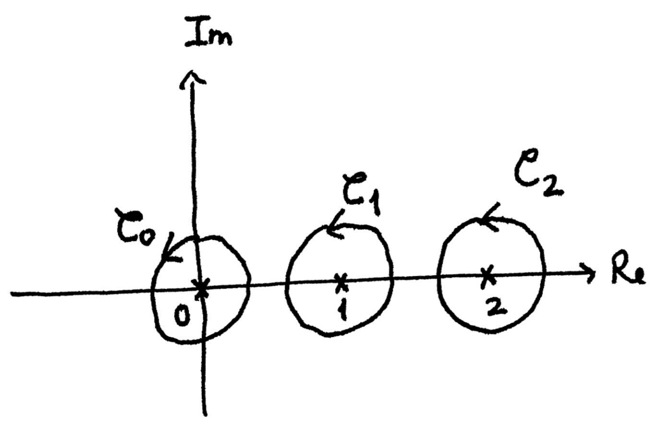
(radius of γ $= r < \text{MIN}\{|\alpha_j - \alpha_k|\}$ $_{\substack{1 \leq k \leq \ell \\ k \neq j}}$)

$$\boxed{A_{j,p} = \frac{1}{2\pi i} \int_\gamma (z - \alpha_j)^{p-1} g(z)\, dz} \; .$$

(19.4) Example.

$$\frac{z^3 - 3}{z(z-1)^2(z-2)^2} = \frac{A}{z} + \frac{B}{z-1} + \frac{C}{(z-1)^2} + \frac{D}{z-2} + \frac{E}{(z-2)^2}$$

Compute C and D.



$$C = \frac{1}{2\pi i} \int_{\mathcal{C}_1} \cancel{(z-1)} \cdot \frac{z^3-3}{z(z-1)^{\cancel{2}}(z-2)^2}\, dz$$

$$= \left[\frac{z^3-3}{z(z-2)^2} \right]_{\text{set } z=1} = \frac{1-3}{1\cdot(-1)^2} = -2.$$

$\boxed{\text{Cauchy's integral formula}}$

$$D = \frac{1}{2\pi i} \int_{\mathcal{C}_2} \frac{z^3-3}{z(z-1)^2(z-2)^2}\, dz = \left[\frac{d}{dz}\left(\frac{z^3-3}{z(z-1)^2} \right) \right]_{\text{set } z=2}$$

$$= \left[\frac{\left(z(z-1)^2\right)(3z^2) - (z^3-3)\left((z-1)^2 + 2z(z-1)\right)}{\left(z(z-1)^2\right)^2} \right]_{\text{set } z=2}$$

$$= \frac{2\cdot(1)^2\cdot 3\cdot(2)^2 - (8-3)\,(1^2 + 2\cdot 2\cdot 1)}{\left(2\cdot(1)^2\right)^2} = \frac{24-25}{4} = \frac{-1}{4}.$$

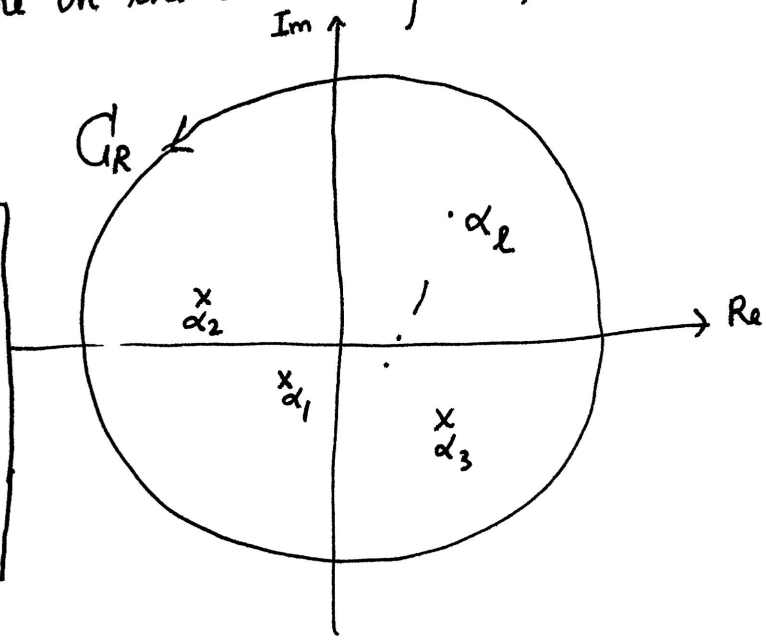
(19.5) $\displaystyle\int_C g(z)\, dz$ for a <u>large enough</u> C.

[Application of
- principle of contour deformation (16.1)
- important inequality (12.9)]

Again : • $g(z) = \dfrac{P(z)}{(z-\alpha_1)^{m_1} \cdots (z-\alpha_\ell)^{m_\ell}}$

• $P(z) = a_n z^n + a_{n-1} z^{n-1} + \dots + a_1 z + a_0$ $\left(\begin{array}{l} n = \deg(P) < m, \\ m = m_1 + \dots + m_\ell \end{array} \right)$

Let C_R be a counterclockwise circle, centered at 0, of radius $R \in \mathbb{R}_{>0}$. Assume $R > \mathrm{MAX}\{|\alpha_1|, \dots, |\alpha_\ell|\}$ (that is, all $\alpha_1, \alpha_2, \dots, \alpha_\ell$ are in the interior of C)



Then

$$\boxed{\frac{1}{2\pi i} \int_{C_R} g(z)\,dz = \begin{cases} 0 & \text{if } n < m-1, \\ a_n & \text{if } n = m-1. \end{cases}}$$

Proof.- Let us consider the 1st case - $n < m-1$.

Note : $\displaystyle\frac{1}{2\pi i} \int_{C_R} g(z)\,dz$ does not depend on R (principle of contour deformation)

Let us estimate its modulus.

• $|P(z)| \leqslant |a_n z^n| + |a_{n-1} z^{n-1}| + \dots + |a_1 z| + |a_0|$ (triangle ineq.)

$= |a_n| R^n + |a_{n-1}| R^{n-1} + \dots + |a_1| R + |a_0|$

$(z \in C_R \ ; \ |z| = R)$.

$$\leq \left(|a_n| + |a_{n-1}| + \dots + |a_0| \right) R^n \quad (\text{assume } R > 1).$$

- for the denominator: $\quad |z - \alpha_K| = |z| \cdot \left| 1 - \frac{\alpha_K}{z} \right|$

$$= R \cdot \left| 1 - \frac{\alpha_K}{z} \right| \geq R \cdot \left(1 - \frac{|\alpha_K|}{R} \right).$$

Take $R > \text{MAX}\{2 \cdot |\alpha_1|, 2 \cdot |\alpha_2|, \dots, 2|\alpha_\ell|\}$.

Then
$$|Q(z)| = |z - \alpha_1|^{m_1} \cdots |z - \alpha_\ell|^{m_\ell}$$
$$\geq R^{m_1 + \dots + m_\ell} \left(1 - \frac{|\alpha_1|}{R} \right)^{m_1} \cdots \left(1 - \frac{|\alpha_\ell|}{R} \right)^{m_\ell}$$
$$\geq R^m \frac{1}{2^m} \qquad (m = m_1 + \dots + m_\ell).$$

Hence,
$$|g(z)| = \frac{|P(z)|}{|Q(z)|} \leq \frac{R^n}{R^m} \cdot 2^m \left(|a_n| + \dots + |a_0| \right)$$

and by important inequality:
$$\left| \frac{1}{2\pi i} \int_{C_R} g(z)\, dz \right| \leq \frac{1}{2\pi} \frac{2^m \left(|a_n| + \dots + |a_0| \right) R^n}{R^m} \cdot 2\pi R$$
$$= \frac{2^m \left(|a_n| + \dots + |a_0| \right)}{R^{m-n-1}} \longrightarrow 0$$

as $R \to \infty$
since $m > n+1$.

$$\Rightarrow \boxed{\frac{1}{2\pi i} \int_{C_R} g(z)\, dz = 0}$$

any γ s.t. $\alpha_1, \dots, \alpha_\ell \in \text{Interior}(\gamma)$

Now assume $n = m-1$. Since $a_n = \frac{1}{2\pi i}\int_{C_R} \frac{a_n}{z}\,dz$,

we get
$$\frac{1}{2\pi i}\int_{C_R}\left(g(z)-\frac{a_n}{z}\right)dz = \frac{1}{2\pi i}\int_{C_R}\frac{zP(z)-a_nQ(z)}{zQ(z)}\,dz$$

$$\boxed{\frac{1}{2\pi i}\int_{C_R} g(z)\,dz - a_n}$$

- $\deg(zQ(z)) = m+1$.
- $\deg(zP(z)-a_nQ(z)) \leq n = m-1$

$\Bigg\}$ – by previous part – we get

$$\frac{1}{2\pi i}\int_{C_R}\frac{zP(z)-a_nQ(z)}{zQ(z)}\,dz = 0.$$

Hence, when $\deg P = \deg Q - 1$, we get

$$\frac{1}{2\pi i}\int_{C_R}\frac{P(z)}{Q(z)}\,dz = a_n \quad (\text{leading coeff. of } P.)$$

$\square$