(19.0) Recall that last time we used Cauchy's theorem and the integral formula to prove

(1) Liouville's Theorem. - A bounded holomorphic function defined on the entire complex plane has to be constant.

(2) Fundamental Theorem of Algebra. - Every polynomial over complex numbers is a product of linear (degree 1) polynomials.

Definition. - A holomorphic function defined on the entire complex plane is called an entire function.

Examples of entire functions:

- Polynomial functions, $e^z$, $\sin(z)$, $\cos(z)$ are entire.

- Rational functions are not entire:

$$g(z) = \frac{P(z)}{Q(z)}; \text{ } P \text{ and } Q \text{ are polynomials},$$

$$\text{Domain of } g = \mathbb{C} - \left\{ z \mid Q(z) = 0 \right\}.$$  \[ \text{finite set - contains at most degree} \text{(Q)} \text{ complex numbers.} \]
(19.1) Partial fraction decomposition of a rational function.

Let \( g(z) = \frac{P(z)}{Q(z)} \) be a rational function with degree \( P = n \)

\[ \text{degree}(Q) = m \]

and assume that \( m > n \).

( Remark: - if degree \( P \) \( \geq \) degree \( Q \), we can perform Euclid's

"long division" algorithm to write \( \frac{P}{Q} = A + \frac{B}{Q} \text{...} \)

polynomial

By Fundamental Theorem of Algebra, we can write \( Q(z) \) as a
product of linear terms. Since \( Q(z) \) may have repeated roots, in
general, this means: there are \( l \) distinct complex numbers,
say \( \alpha_1, \alpha_2, \ldots, \alpha_l \in \mathbb{C} \) and \( l \) positive integers

\( m_1, m_2, \ldots, m_l \in \mathbb{Z}_{\geq 1} \)

such that \( Q(z) = c (z - \alpha_1)^{m_1} (z - \alpha_2)^{m_2} \ldots (z - \alpha_l)^{m_l} \)

- \( c \in \mathbb{C}, c \neq 0 \), is the coefficient of \( z^m \) in \( Q(z) \).
- \( \text{degree}(Q) = m = m_1 + m_2 + \ldots + m_l \).
- \( m_j \) is called the multiplicity of the root \( \alpha_j \) of \( Q(z) \).

\( 1 \leq j \leq l \).

We are also going to rescale \( c = 1 \). (without loss of generality).
After all this preparation, our rational function takes the following form:

\[ g(z) = \frac{P(z)}{(z-\alpha_1)^{m_1} \cdots (z-\alpha_k)^{m_k}} \]  

\[ \text{degree}(P) < m \]  

\[ m = m_1 + \ldots + m_k \]

**Partial Fractions decomposition of** \( g(z) \):

\[
\begin{align*}
g(z) &= \frac{A_{1,1}}{z-\alpha_1} + \frac{A_{1,2}}{(z-\alpha_1)^2} + \ldots + \frac{A_{1,m_1}}{(z-\alpha_1)^{m_1}} \\
&+ \frac{A_{2,1}}{z-\alpha_2} + \frac{A_{2,2}}{(z-\alpha_2)^2} + \ldots + \frac{A_{2,m_2}}{(z-\alpha_2)^{m_2}} \\
&+ \ldots + \frac{A_{k,1}}{z-\alpha_k} + \frac{A_{k,2}}{(z-\alpha_k)^2} + \ldots + \frac{A_{k,m_k}}{(z-\alpha_k)^{m_k}} \\
\end{align*}
\]

(here, \( A_{i,j}, \ldots, A_{k,m_k} \in \mathbb{C} \).

(19.2) **Example.** Let \( g(z) = \frac{z^2 + 2}{(z-1)(z+1)^2} \). 

\[
\begin{bmatrix}
\text{so, } \alpha_1 = 1, m_1 = 1 \\
\alpha_2 = -1, m_2 = 2 \\
m = 1 + 2 = 3 \\
> \text{deg (Numerator) = 2.}
\end{bmatrix}
\]

\[
g(z) = \frac{z^2 + 2}{(z-1)(z+1)^2} = \frac{A}{z-1} + \frac{B}{z+1} + \frac{C}{(z+1)^2}.
\]

**Problem.** Compute \( A, B \) and \( C \).
Solution.  - Clear the denominator:

\[ z^2 + 2 = A(z+1)^2 + B(z-1)(z+1) + C(z-1). \]

- Set \( z = 1 \) to get: \( 3 = A \cdot 4 \Rightarrow A = \frac{3}{4} \).
- Set \( z = -1 \) to get: \( 3 = C \cdot 2 \Rightarrow C = \frac{3}{2} \).

So, \( z^2 + 2 = \frac{3}{4}(z+1)^2 + B(z^2-1) + \frac{3}{2}(z-1) \). Compare coefficients of \( z^2 \):

\[
1 = \frac{3}{4} + B \Rightarrow B = \frac{1}{4}.
\]

Hence,

\[
\frac{z^2 + 2}{(z-1)(z+1)^2} = \frac{3/4}{z-1} + \frac{1/4}{z+1} + \frac{3/2}{(z+1)^2}.
\]

(19.3) Using Cauchy’s integral formula to compute partial fractions.

As an easy application of Cauchy’s integral formula (or, by a quick and equally easy direct verification), we have

\[
\frac{1}{2\pi i} \int \frac{1}{(z-\alpha)^{k}} \, dz = \begin{cases} 1 & \text{if } k = 1 \\ 0 & \text{if } k \neq 1 \end{cases}.
\]

(\( \gamma(\theta) = \alpha + r \cdot e^{i\theta} \) (0 \( \leq \theta \leq 2\pi))

\( \gamma = \text{circle of radius } r \in \mathbb{R}_0^+ \)
Thus, in the partial fractions decomposition (*) of page 3 above, if we want to compute, say $A_{j,p}$:

\[ 1 \leq j \leq l \; ; \; 1 \leq p \leq m_j \]

- multiply both sides of (*) by $(z - \alpha_j)^{p-1}$.
- integrate both sides:
  \[ \frac{1}{2\pi i} \int_{\gamma} \frac{g(z)}{(z - \alpha_j)^p} \, dz \]
  where $\gamma$ is a small circle (counterclockwise) centered at $\alpha_j$ (small enough so that no other $\alpha_k$ is in the interior of $\gamma$).

\[
A_{j,p} = \frac{1}{2\pi i} \int_{\gamma} \frac{g(z)}{(z - \alpha_j)^p} \, dz
\]

(19.4) Example.

\[
\frac{z^3 - 3}{z(z-1)^2(z-2)^2} = \frac{A}{z} + \frac{B}{z-1} + \frac{C}{(z-1)^2} + \frac{D}{z-2} + \frac{E}{(z-2)^2}
\]

Compute $C$ and $D$. 

[Diagram with contours $C_0$, $C_1$, and $C_2$]
\[ C = \frac{1}{2\pi i} \int_{C_1} \frac{Z^3 - 3}{Z(Z-1)^2(Z-2)} \, dZ \]

\[ \Rightarrow \left[ \frac{Z^3 - 3}{Z(Z-2)^2} \right]_{\text{Set } Z=1} = \frac{1-3}{1 \cdot (-1)^2} = -2. \]

Cauchy's integral formula

\[ D = \frac{1}{2\pi i} \int_{C_2} \frac{Z^3 - 3}{Z(Z-1)^2(Z-2)} \, dZ \equiv \left[ \frac{d}{dZ} \frac{Z^3 - 3}{Z(Z-1)^2} \right]_{\text{Set } Z=2} \]

\[ = \left[ \frac{(Z(Z-1)^2)(3Z^2) - (Z^3 - 3)(Z-1)^2 + 2Z(Z-1)}{(Z(Z-1)^2)^2} \right]_{\text{Set } Z=2} \]

\[ = \frac{2 \cdot (1)^2 \cdot 3 \cdot (2)^2 - (8-3)(1^2 + 2 \cdot 2 \cdot 1)}{(2 \cdot (1)^2)^2} = \frac{24 - 25}{4} = \frac{-1}{4}. \]

(19.5) \[ \int_{C} g(Z) \, dZ \quad \text{for a large enough } C. \]

[Application of  
\[ \cdot \text{ principle of contour deformation (16.1)} \]
\[ \cdot \text{ important inequality (12.9)} \] ]
Again: \[ g(z) = \frac{P(z)}{(z-\alpha_1)^{m_1} \cdots (z-\alpha_L)^{m_L}} \]

- \[ P(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0 \quad (n = \deg(P) < m, \quad m = m_1 + \cdots + m_L) \]

Let \( C_R \) be a counterclockwise circle, centered at 0, of radius \( R \in \mathbb{R}_{>0} \). Assume \( R > \max \{ |\alpha_1|, \ldots, |\alpha_L| \} \) (that is, all \( \alpha_1, \alpha_2, \ldots, \alpha_L \) are in the interior of \( C \))

\[
\frac{1}{2\pi i} \int_{C_R} g(z) dz = \begin{cases} 
0 & \text{if } n < m-1, \\
 a_n & \text{if } n = m-1.
\end{cases}
\]

Proof: Let us consider the 1st case - \( n < m-1 \).

Note: \( \frac{1}{2\pi i} \int_{C_R} g(z) dz \) does not depend on \( R \) (principle of contour deformation).

Let us estimate its modulus.

- \( |P(z)| \leq |a_n z^n| + |a_{n-1} z^{n-1}| + \cdots + |a_1 z| + |a_0| \) (triangle ineq.)

\[
= |a_n| R^n + |a_{n-1}| R^{n-1} + \cdots + |a_1| R + |a_0|
\]

(\( z \in C_R \; ; \; |z| = R \)).
\[ \leq (|a_n| + |a_{n-1}| + \ldots + |a_1|) R^n \quad (\text{assume } R > 1). \tag{8} \]

- For the denominator:
  \[ |z - \alpha_k| = |z| \left| 1 - \frac{\alpha_k}{z} \right| \]
  \[ = R \left| 1 - \frac{\alpha_k}{z} \right| \geq R \left( 1 - \frac{|\alpha_k|}{R} \right). \]

Take \( R > \max \{2 |\alpha_1|, 2 |\alpha_{21}|, \ldots, 2 |\alpha_{e1}| \} \).

Then
\[ |Q(z)| = |z - \alpha_1|^{m_1} \ldots |z - \alpha_e|^{m_e} \]
\[ \geq R^{m_1 + \ldots + m_e} \left( 1 - \frac{|\alpha_1|}{R} \right)^{m_1} \ldots \left( 1 - \frac{|\alpha_e|}{R} \right)^{m_e} \]
\[ \geq R^m \frac{1}{2^m} \quad (m = m_1 + \ldots + m_e). \]

Hence,
\[ |g(z)| = \frac{|P(z)|}{|Q(z)|} \leq \frac{R^n}{R^m} 2^m (|a_n| + \ldots + |a_1|) \]

and by important inequality:
\[ \frac{1}{2\pi i} \oint_{C_R} g(z) \, dz \leq \frac{1}{2\pi} \frac{2^m (|a_n| + \ldots + |a_1|) R^n}{R^m} \cdot 2\pi R \]
\[ = \frac{2^m (|a_n| + \ldots + |a_1|)}{R^{m-n-1}} \rightarrow 0 \quad \text{as } R \rightarrow \infty \]
since \( m > n+1 \).

\[ \frac{1}{2\pi i} \int_{C_R} g(z) \, dz = 0 \]

for any \( g \) s.t. \( \alpha_1, \ldots, \alpha_e \in \text{Interior}(R) \).
Now assume $n = m - 1$. Since $a_n = \frac{1}{2\pi i} \int_{C_R} \frac{a_n}{z} \, dz$, we get

\[
\frac{1}{2\pi i} \int_{C_R} \left( g(z) - \frac{a_n}{z} \right) \, dz = \frac{1}{2\pi i} \int_{C_R} \frac{z P(z) - a_n Q(z)}{z Q(z)} \, dz.
\]

- $\deg(z Q(z)) = m + 1$.
- $\deg(z P(z) - a_n Q(z)) \leq n = m - 1$. 

By previous part, we get

\[
\frac{1}{2\pi i} \int_{C_R} \frac{z P(z) - a_n Q(z)}{z Q(z)} \, dz = 0.
\]

Hence, when $\deg P = \deg Q - 1$, we get

\[
\frac{1}{2\pi i} \int_{C_R} \frac{P(z)}{Q(z)} \, dz = a_n \quad \text{(leading coeff. of P.)}
\]