Lecture 20

(20.0) Today we are going to study further applications of Cauchy's Theorems.

- Maximum modulus principle.
- Schwarz' Lemma.
- Classification of holomorphic re-parametrizations of a disc.

We will begin by proving a useful lemma - using Cauchy-Riemann equations.

(20.1) Lemma. Let $f:\Omega \to \mathbb{C}$ be a holomorphic function, defined on an open set $\Omega \subseteq \mathbb{C}$. [$\Omega$ is assumed to be connected.]

(a) If $\overline{f}(z)$ is also holomorphic, then f is constant.

(b) If $|f(z)| = c$ (constant) for every $z \in \Omega$, then f is constant.

Proof. (a) Let us write $f(z) = u(x,y) + i\,v(x,y)$ $\left(\begin{array}{l} x = Re(z) \\ y = Im(z) \end{array}\right)$.

f being holomorphic, we have the Cauchy-Riemann equations

$$u_x = v_y \quad \text{and} \quad u_y = -v_x \quad - (1)$$

$\overline{f}(z) = u(x,y) - i\,v(x,y)$ being holomorphic, we get (C-R eq^{n}s for $(u,-v)$):

$$u_x = -v_y \quad \text{and} \quad u_y = +v_x \quad - (2)$$

Combining (1) and (2), we get: $u_x = 0$, $u_y = 0$, $v_x = 0$, $v_y = 0$.

Hence $f'(z) = 0$ for every $z \in \Omega \Rightarrow f$ is constant.

(b). Assume $|f(z)| = c \in \mathbb{R}_{\geq 0}$ for every $z \in \Omega$. If $c = 0$, then $f(z) = 0$ for every $z \in \Omega$. Assume $c > 0$. Then ($f(z) \neq 0 \; \forall z \in \Omega$):

$$f(z)\,\overline{f(z)} = c^2 \Rightarrow \overline{f}(z) = \frac{c^2}{f(z)} \text{ for every } z \in \Omega.$$

is again holomorphic. By (a) then f is a constant function. □

(20.2) Maximum modulus principle.

Lemma. - Let $\alpha \in \mathbb{C}$ and $R \in \mathbb{R}_{>0}$. Assume that $f(z)$ is a holomorphic function defined on the open disc: $D(\alpha; R) = \{z \in \mathbb{C} \mid |z - \alpha| < R\}$.

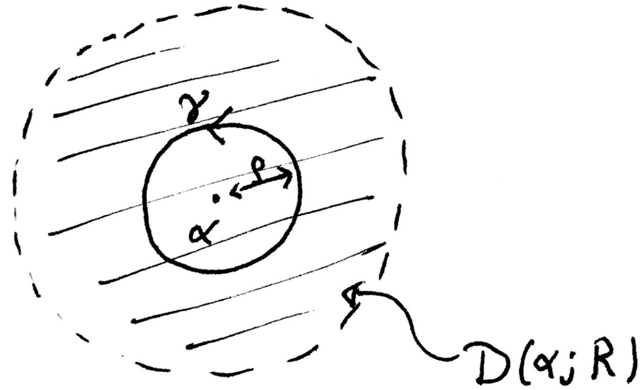
$$f : D(\alpha; R) \to \mathbb{C}.$$

If $|f(\alpha)| \geq |f(z)|$ for every $z \in D(\alpha; R)$, then f is a constant function.

Proof. - Let ρ be any positive real number less than R. $(0 < \rho < R)$. Let $\gamma : [0, 2\pi] \to D(\alpha; R)$

$$\gamma(\theta) = \rho \cdot e^{i\theta} + \alpha$$

be the counterclockwise circle of radius ρ, centred at α.



By Cauchy's integral formula

$$f(\alpha) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(z)}{z-\alpha}\, dz$$

$$= \frac{1}{2\pi i} \int_0^{2\pi} \underbrace{\boxed{\frac{f(\alpha+\rho e^{i\theta})}{\rho \cdot e^{i\theta}}}}_{\frac{f(z)}{z-\alpha} \text{ at } z=\gamma(\theta)} \cdot \underbrace{\rho \cdot i \cdot e^{i\theta}}_{\gamma'(\theta)}\, d\theta$$

$$\Rightarrow \quad \boxed{f(\alpha) = \frac{1}{2\pi} \int_0^{2\pi} f(\alpha+\rho e^{i\theta})\, d\theta}$$

Remark. - This expression (see Problem #5 of Set 5) is often referred to as "Gauss' mean value theorem". It is interpreted as saying: <u>the value of f at α is the average of its values on the circle γ</u>.

Therefore we get

$$|f(\alpha)| \leq \frac{1}{2\pi}\int_0^{2\pi} |f(\alpha+\rho e^{i\theta})|\, d\theta$$

$$\leq \frac{1}{2\pi}\int_0^{2\pi} |f(\alpha)|\, d\theta \quad \left(\text{using the hypothesis: } |f(\alpha)| \geq |f(z)| \text{ for every } z \in D(\alpha;R).\right)$$

$$= |f(\alpha)|.$$

Hence all the inequalities above have to be equalities:

$$|f(\alpha)| = \frac{1}{2\pi}\int_0^{2\pi} |f(\alpha+\rho e^{i\theta})|\, d\theta.$$

$$\Rightarrow \quad \frac{1}{2\pi}\int_0^{2\pi} \Big(|f(\alpha)| - |f(\alpha+\rho e^{i\theta})|\Big)\, d\theta = 0.$$

But $\boxed{\theta \longmapsto |f(\alpha)| - |f(\alpha+\rho e^{i\theta})|}$ is a continuous ; ≥ 0 function of $\theta \in [0, 2\pi]$.

Its integral being zero implies that this function is identically 0.

That is, $|f(\alpha)| = |f(\alpha+\rho e^{i\theta})|$; for every $0 \leq \theta \leq 2\pi$.

Since $0 < \rho < R$ was picked arbitrarily in the beginning, we conclude that $|f(\alpha)| = |f(z)|$ for every $z \in D(\alpha; R)$.

Thus $f: D(\alpha; R) \to \mathbb{C}$ is a holomorphic function of constant modulus. By Lemma (20.1) (b) above, f has to be constant. $\square$

(20.3) Remarks - (1) Later in this course (using Taylor series) we will prove a very strong property of holomorphic functions:

> If $f_1, f_2 : \Omega \to \mathbb{C}$ are two holomorphic functions, defined on an open, connected subset $\Omega \subseteq \mathbb{C}$, such that there is an open disc $D(\alpha; r) \subset \Omega$ where f_1 and f_2 agree (i.e. $\alpha \in \Omega$; $r \in \mathbb{R}_{>0}$ so that $D(\alpha; r) \subset \Omega$; and $f_1(z) = f_2(z)$ $\forall\, z \in D(\alpha; r)$)
>
> Then $f_1(z) = f_2(z)$ for every $z \in \Omega$.

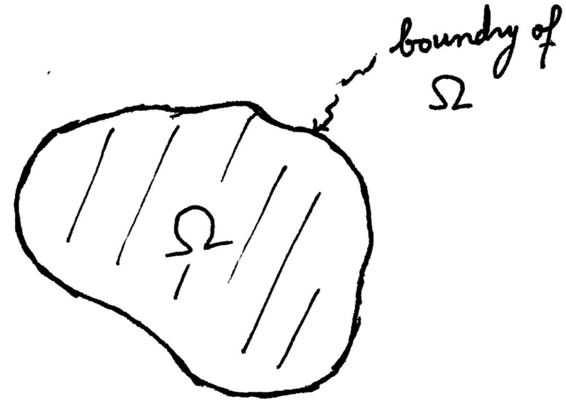
This will allow us to strengthen Lemma (20.2) above to the following statement:

Theorem. If $f : \Omega \to \mathbb{C}$ is a non-constant, holomorphic function, defined on an open, connected set $\Omega \subseteq \mathbb{C}$, then $|f|$ has NO LOCAL MAXIMA in Ω.

(LOCAL MAX for $|f|$ is an $\alpha \in \Omega$ for which there is some $r \in \mathbb{R}_{>0}$ so that
- $D(\alpha; r) \subset \Omega$
- $|f(\alpha)| \geq |f(z)|$ $\forall\, z \in D(\alpha; r)$.)

(2) Assume that $f : \Omega \to \mathbb{C}$ is a holomorphic function defined on a bounded open, connected set $\Omega \subseteq \mathbb{C}$.
Further assume that f extends to the boundry of Ω.

Thus $|f| : \overline{\Omega} \to \mathbb{R}_{\geq 0}$ is a continuous function, defined on a closed and bounded set. Hence, it must attain its max. value somewhere. Lemma (20.2) - and its extension-Theorem from page 5 - imply that this MAX can only occur on the boundry, unless f is just a constant function.



$\overline{\Omega} = \Omega \cup (\text{boundry of } \Omega)$ is closed and bounded.

(20.4) Schwarz' Lemma.

Let $f : D(0;1) \to \mathbb{C}$ be a holomorphic function such that:

$\{z \mid |z| < 1\}$

(i) $f(0) = 0$,

(ii) $|f(z)| \leq 1$ for every $z \in D(0;1)$.

Then $|f(z)| \leq |z|$ for every $z \in D(0;1)$. If, moreover,

$|f(z_0)| = |z_0|$ for some $z_0 \in D(0;1)$, then there exists $\theta \in \mathbb{R}$ such that $f(z) = e^{i\theta} \cdot z \quad \forall\, z \in D(0;1)$.

(i.e. $f(z)$ = rotate z by angle θ)

Proof. Consider the function
$$g(z) = \begin{cases} \dfrac{f(z)}{z} & ; \ z \neq 0, \\ f'(0) & ; \ z = 0. \end{cases}$$

g is again a holomorphic function (check this!) and for z_0 on the boundry of $D(0;1)$, i.e. $|z_0| = 1$, we have

$$\lim_{\substack{z \to z_0 \\ (z \in D(0;1))}} |g(z)| = \lim_{\substack{z \to z_0 \\ (z \in D(0;1))}} |f(z)| \leq 1.$$

$\{z \mid |z| \leq 1\}$

Hence the absolute max. of $|g| : \overline{D(0,1)} \to \mathbb{R}_{>0}$, which must occur at the boundry – by Remark (2) above, is ≤ 1.

$\Rightarrow \quad |f(z)| \leq |z|$ for every $z \in D(0;1)$.

If $|g(z_0)| = 1$ for some $z_0 \in D(0;1)$, then again by Lemma (20.2), $g(z) = c$ (constant), $|c| = 1$ i.e. $c = e^{i\theta}$ for some $\theta \in \mathbb{R}$.

That is, $f(z) = e^{i\theta} z \quad \forall\, z \in D(0;1)$ as claimed. $\square$

(20.5)* Classification of holomorphic automorphisms of a disc.

Let $D = D(0;1)$. A <u>holomorphic automorphism</u> of D is a <u>holomorphic function</u> $f : D \to D \subset \mathbb{C}$, <u>which admits a holomorphic inverse</u>: i.e., there is $g : D \to D$ (holomorphic) such that $f(g(z)) = z = g(f(z))$. (Such f's are also called holomorphic re-parametrizations of the disc $D(0;1)$.)

Theorem. (1) Let $a \in D(0;1)$ (i.e. $a \in \mathbb{C}$, $|a| < 1$).

Define $f_a(z) = \dfrac{z-a}{1-\bar{a}z}$. Then f_a is a holomorphic automorphism of D.

(2) Let $f : D \to D$ be any holomorphic automorphism. Then there exists $a \in D$ and $\theta \in \mathbb{R}$ such that

$$f(z) = e^{i\theta} \cdot \frac{z-a}{1-\bar{a}z} \quad \left(= e^{i\theta} f_a(z)\right).$$

Proof of (1): $f_a(z)$ is defined on the disc $D(0; \frac{1}{|a|})$ which contains $\overline{D(0;1)}$, since $|a| < 1$.

For z_0 on the boundry of $D(0;1)$, i.e. $|z_0| = 1$, we get

$$|f_a(z_0)| = \frac{|z_0 - a|}{\underbrace{|z_0(\bar{z}_0 - \bar{a})|}_{\text{(replace 1 by } z_0\bar{z}_0)}} = \frac{|z_0 - a|}{|\overline{z_0 - a}|} = 1.$$

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*Optional

Hence, by maximum modulus principle,

$$|f_a(z)| < 1 \quad \text{for every } z \in D(0;1)$$

it cannot be $= 1$, since that would mean f_a is constant.

i.e., $f_a : D \to D$.

Now $f_a(f_{-a}(z)) = z = f_{-a}(f_a(z)) \Rightarrow f_a$ is a holomorphic automorphism.

(easy exercise).

Proof of (2). Let $f : D \to D$ be any holomorphic automorphism.

Let $b = f(0) \in D$. Set $F : D \to D$ to be

$$F(z) = f_b(f(z)).$$

Then $F(0) = 0$ and by Schwarz' Lemma $\boxed{|F(z)| \leq |z|. \ (\forall z \in D)}$

If $G : D \to D$ is the inverse of F, Schwarz' Lemma applies to G as well, and we get $|G(w)| \leq |w|$ for every $w \in D$.

Put $w = F(z)$ in this inequality to get $\boxed{|z| \leq |F(z)|}$

Hence $|F(z)| = |z| \ \forall z \in D$, and again by Schwarz' Lemma,

$F(z) = e^{i\theta} \cdot z$. As $F(z) = f_b(f(z))$, we get:

$$f(z) = e^{i\theta} f_a(z) \quad \text{where } a = -e^{-i\theta} b.$$

□