Lecture 21

Sequences and series

(21.0) In the next (third) part of the course, we will be dealing with series of holomorphic functions:

$u_0(z), u_1(z), u_2(z), \dots$ holomorphic functions defined on an open set $\Omega \subseteq \mathbb{C}$. $\rightsquigarrow$ $u_0(z) + u_1(z) + u_2(z) + \dots$ (notation: $\sum_{k=0}^{\infty} u_k(z)$)

Today's lecture is going to be a review of <u>convergence</u> of a sequence or series of numbers (real or complex), and of functions (of real/complex variables).

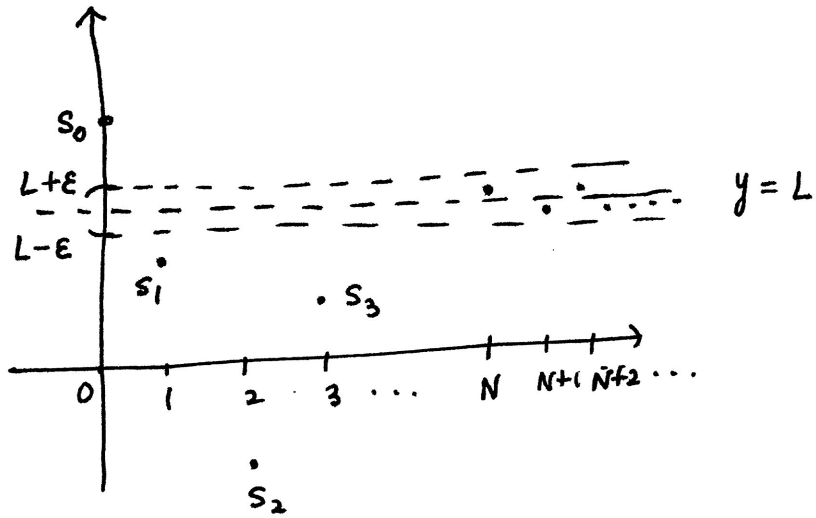
(21.1) Definition of convergence and limit.

Let $s_0, s_1, s_2, \dots$ be a sequence of numbers (real or complex)

We say $\boxed{\lim_{n \to \infty} s_n = L}$ if given any $\varepsilon > 0$, we can find $N > 0$ such that:

$$|s_n - L| < \varepsilon \quad \text{for every } n \geq N.$$

<u>Picture</u>: $s_0, s_1, \ldots$ are real - plotted on x-y plane as points (n, s_n)



$\lim_{n\to\infty} s_n = L$ signifies

that - after $s_0, \ldots, s_{N-1}$;

all numbers $\{s_N, s_{N+1}, \ldots\}$ lie in the band $(L-\varepsilon, L+\varepsilon)$ sketched above.

<u>Cauchy's criterion.</u> A sequence of numbers $s_0, s_1, s_2, \ldots$ is <u>convergent</u> if given any $\varepsilon > 0$, we can find $N > 0$ such that:

$$|s_n - s_m| < \varepsilon \quad \text{for every } n, m > N.$$

Remarks. - (1) Both these definitions are attributed to Augustin-Louis Cauchy (1789 - 1857)*. The second one has the advantage over the first as <u>the explicit limit L need not be known</u>.

(2) The fact that a <u>convergent sequence of numbers $s_0, s_1, \ldots$ always has a unique limit</u> is proved using the existence of lim sup and lim-inf of real numbers (often called "the <u>completeness axiom</u> of real numbers"). I will recall it

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* Analyse algébrique (1821)

below, but the proof of $\boxed{\begin{array}{c} s_0, s_1, s_2, \dots \\ \text{convergent} \end{array}} \Rightarrow \boxed{\begin{array}{c} s_0, s_1, s_2, \dots \\ \text{has a unique limit} \end{array}}$

will be given only in Optional Reading. For now, we will take it as true.

(21.2)* Lim-sup / lim-inf and completeness axiom of $\mathbb{R}$.

Let $x_0, x_1, x_2, \dots$ be a sequence of <u>real</u> numbers.

$\ell = \limsup x_n$ (also denoted by $\overline{\lim}\, x_n$) means :

(i) For every $\varepsilon > 0$, there exists $N > 0$ such that

$$x_n < \ell + \varepsilon \quad \text{for all } n \geq N.$$

(ii) For every $\varepsilon > 0$ and every $n > 0$; there exists some $k > n$

such that $\quad x_k > \ell - \varepsilon$.

Thus, we want - for every $\varepsilon > 0$ - there to be infinitely many x_n's in the interval $(\ell - \varepsilon, \ell + \varepsilon)$. Moreover - almost all of x_n's should be less than $\ell + \varepsilon$.

[e.g. $s_0 = 1,\ s_1 = -1,\ s_2 = 1,\ s_3 = -1,\ \dots,\ s_n = (-1)^n, \dots$

$\begin{cases} \overline{\lim}\ s_n = 1 \\ \underline{\lim}\ s_n = -1 \end{cases}$. One analogously defines lim-inf, or $\underline{\lim}$,]

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*optional

Completeness axiom of $\mathbb{R}$. — any sequence of real numbers which is bounded from above (resp. bounded from below) has a lim-sup (resp. lim-inf).

(21.3) Sum of an infinite series.

Let $a_0, a_1, a_2, \ldots$ be numbers (real or complex)

By $\boxed{S = \sum_{k=0}^{\infty} a_k}$ we mean $\boxed{S = \lim_{n\to\infty} \sum_{k=0}^{n} a_k}$.

That is, form a sequence of numbers:

$$s_0 = a_0 \; ; \quad s_1 = a_0 + a_1 \; ; \quad s_2 = a_0 + a_1 + a_2 \; ; \; \ldots, \; s_n = \sum_{k=0}^{n} a_k \; ; \ldots$$

and $S = \lim_{n\to\infty} s_n$ as per Definition (21.1) above

Unfolding the definition :– Given any $\varepsilon > 0$, we can find $N > 0$ such that $|s_n - S| < \varepsilon$ for every $n \geq N$.

Note : $S - s_n = \sum_{k=n+1}^{\infty} a_k$. So, this definition, in plain English, means that for any given $\varepsilon > 0$ (how accurate you want the sum to be?) there is a finite number N such that $S = a_0 + a_1 + \ldots + a_N$ with an accuracy of ε.

According to the Cauchy criterion for convergence:

$\sum_{k=0}^{\infty} a_k$ is <u>convergent</u>, if for any given $\varepsilon > 0$, we can find $N > 0$, so that

$$\left| \sum_{k=n+1}^{m} a_k \right| < \varepsilon \quad \text{for every} \quad \begin{array}{l} n < m \\ N \leq n. \end{array}$$

Note: in terms of sequence
$S_0 = a_0$
$S_1 = a_0 + a_1$
$S_2 = a_0 + a_1 + a_2$
$\dots$

$$S_m - S_n = \sum_{k=n+1}^{m} a_k$$

A series which is not convergent is called <u>divergent</u>.

(21.4) Convergence tests from Calculus II.

(i) Ratio test (d'Alembert 1768).

If there is a positive number $\rho < 1$ and $N > 0$, such that $\left| \frac{a_{n+1}}{a_n} \right| < \rho$ for every $n \geq N$.

then $\sum_{k=0}^{\infty} a_k$ is convergent.

Examples : (1) Let $z \in \mathbb{C}$, $|z| < 1$. Then

$$\sum_{k=0}^{\infty} z^k \quad \text{is convergent.}$$

(2) $1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots$ is convergent.

(ii) Root test (Cauchy 1821).

If $\boxed{\limsup |a_n|^{\frac{1}{n}} < 1}$, then $\sum_{k=0}^{\infty} a_k$ is convergent.

Example of a divergent series : $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$ is divergent.

Proof - For any $n \geq 1$, we have

$$\frac{1}{n} + \frac{1}{n+1} + \cdots + \frac{1}{2n} > \underbrace{\frac{1}{2n} + \frac{1}{2n} + \cdots + \frac{1}{2n}}_{n\text{-terms}} = \frac{1}{2}.$$

So, take $\varepsilon > 0$ to be anything less than $\frac{1}{2}$. We will never be able to find N that makes the following statement true:

$$\boxed{\left| \frac{1}{n} + \frac{1}{n+1} + \cdots + \frac{1}{m} \right| < \varepsilon \quad \text{for every } m > n \geq N}$$

Since no matter what n is, taking $m = 2n$, gives us

$$\frac{1}{2} < \left| \frac{1}{n} + \frac{1}{n+1} + \cdots + \frac{1}{2n} \right|, \text{ and } \varepsilon < \frac{1}{2}. \qquad \square$$

Exercise. — Show that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^a}$ is convergent,

for any $a > 1$.

(21.5) Sequences and series of functions.

Let us consider a sequence of functions $f_0(x), f_1(x), f_2(x), \dots$

all defined on an open set Ω (of $\mathbb{R}$, or of $\mathbb{C}$).

- Pointwise convergence. If for any fixed $x_0 \in \Omega$, the sequence of numbers $f_0(x_0), f_1(x_0), \dots$ converges, we can define f by $\boxed{f(x_0) = \lim_{n\to\infty} f_n(x_0)}$.

We say $\{f_n\}_{n=0}^{\infty}$ converges to f pointwise.

Remark. — (i) The same terminology applies to a series of functions: $\displaystyle u(x) = \sum_{k=0}^{\infty} u_k(x)$ pointwise, means:

- we are given functions $u_0(x), u_1(x), \dots$
- we form a sequence $f_0(x) = u_0(x)$; $f_1(x) = u_0(x) + u_1(x)$; ...

$$u(x) = \lim_{n\to\infty} f_n(x) \quad \text{pointwise.}$$

(ii) Pointwise convergence <u>does not</u> guarentee that the properties of the functions (e.g. continuity, differentiability etc.) will remain true at the limit. For example, consider the series of functions defined on $\mathbb{R}$:

$$x^2 + \frac{x^2}{1+x^2} + \frac{x^2}{(1+x^2)^2} + \frac{x^2}{(1+x^2)^3} + \cdots$$

$$f_n(x) = x^2\left(1 + \frac{1}{1+x^2} + \cdots + \frac{1}{(1+x^2)^n}\right) \qquad \text{(sum of first } n \text{ terms)}$$

$$= x^2 \, \frac{1 - \dfrac{1}{(1+x^2)^{n+1}}}{1 - \dfrac{1}{1+x^2}} = 1 + x^2 - \frac{1}{(1+x^2)^n}$$

For $x \neq 0$; $\lim\limits_{n\to\infty} f_n(x) = 1 + x^2$

For $x = 0$; each $f_n(x) = 0$; hence $\lim\limits_{n\to\infty} f_n(x) = 0$.

Hence $\underbrace{f_n}_{\text{all continuous}} \longrightarrow \underbrace{f}_{\text{not continuous}}$ as $n \to \infty$ <u>pointwise</u>. where $f(x) = \begin{cases} 1 + x^2 ; & x \neq 0 \\ 0 ; & x = 0 \end{cases}$

The notion of <u>uniform convergence</u> is needed to fix this.
(Weierstrass 1815-1897)

(21.6) Again, let $f_0(x), f_1(x), \ldots$ be functions defined on an open set Ω. Let $A \subset \Omega$ and $f(x)$ another function.

$\boxed{\lim_{n\to\infty} f_n = f \text{ uniformly on } A}$ (or with respect to A)

means : given any $\varepsilon > 0$, we can find $N > 0$ such that

$$\boxed{\begin{array}{ll} |f_n(x) - f(x)| < \varepsilon & \text{for every } n \geq N \\ & \underline{\text{and}} \text{ for every } x \in A \end{array}}$$

(<u>Important</u> : This N should "work" for all points in A simultaneously.)

$\boxed{\lim_{n\to\infty} f_n = f \text{ uniformly}}$ (with no reference to a subset $A \subset \Omega$)

means — for every <u>compact (i.e. closed and bounded)</u> subset $A \subset \Omega$

$$\lim_{n\to\infty} f_n = f \text{ uniformly w.r.t. } A.$$

Later we will prove that uniform convergence preserves all the nice properties (e.g. f_n : holomorphic ; $\lim_{n\to\infty} f_n = f$
$\Rightarrow f$ is holomorphic, and $f' = \lim_{n\to\infty} f_n'$).