COMPLEX ANALYSIS: LECTURE 22

(22.0) What is in this lecture.–

(1) Definition of uniform convergence of a sequence of functions (§22.1), and a series of functions (§22.2). Why is it relevant - Weierstrass' theorem (§22.6).
(2) Definition of a power series and its radius of convergence - Abel's theorem (§22.3).
(3) How to compute the radius of convergence of a given power series (§22.4) and examples (§22.5).
(4) Proofs (optional) of Weierstrass' theorem (§22.7) and Abel's theorem (§22.8).

(22.1) Uniform convergence.– Let $\Omega \subset \mathbb{C}$ be an open set. Let $f_n : \Omega \to \mathbb{C}$ $(n = 0, 1, 2, \ldots)$ be a sequence of functions. Recall (Lecture 21, page 7) that when we say $\{f_n\}_{n=0}^{\infty}$ converges *pointwise*, we mean that for every $z_0 \in \Omega$, the resulting sequence of numbers $\{f_n(z_0)\}_{n=0}^{\infty}$ is convergent. Its limit defines a new function f for us via the rule: $\boxed{f(z_0) = \lim_{n\to\infty} f_n(z_0)}$. In Lecture 21, page 8, we saw that pointwise convergence does not respect the desirable properties (like continuity) of functions. For that we need the notion of uniform convergence.

Definition. We say that $\{f_n\}_{n=0}^{\infty}$ converges *uniformly*, if for every *compact* set $K \subset \Omega$, and $\varepsilon > 0$, we can find an $N > 0$ such that the following statement holds true:

$$\text{for every } n, m \geq N \text{ and } z \in K, \text{ we have } |f_n(z) - f_m(z)| < \varepsilon.$$

Remarks. (1) Recall that a subset $K \subset \mathbb{C}$ is said to be compact, if it is both closed and bounded (see Lecture 3, page 8).

(2) A singleton $\{z_0\}$ is clearly compact. Thus, uniform convergence implies pointwise convergence, and we obtain a new function $f : \Omega \to \mathbb{C}$, via the rule: $\boxed{f(z_0) = \lim_{n\to\infty} f_n(z_0)}$. We will often write in words *f_n converges uniformly to f, as $n \to \infty$*, or in symbols $\boxed{\lim_{n\to\infty} f_n = f}$ *uniformly.*

(3) In the definition above, the number $N > 0$ will depend on ε and K. The point is that it works *for all* $z_0 \in K$ at the same time. This is where the notion of uniform convergence differs from that of pointwise convergence.

(22.2) Series of functions.– As we saw in Lecture 21 (§21.3, page 4) the notions of convergence (pointwise, or uniform) for a *series* of functions are the same as those for the associated sequence of finite sums. This means the following.

Given an infinite series $u_0(z)+u_1(z)+\cdots$ of functions, abbreviated as $\sum_{k=0}^{\infty} u_k(z)$, consider the following sequence of functions:

$$f_0(z)=u_0(z),\ f_1(z)=u_0(z)+u_1(z),\ \ldots,\ \boxed{f_n(z)=\sum_{k=0}^{n} u_k(z)},\ \ldots$$

Then $\sum_{k=0}^{\infty} u_k(z)$ is uniformly convergent if the sequence $\{f_n(z)\}_{n=0}^{\infty}$ is.

To spell it out again: in order to ensure that $\sum_{k=0}^{\infty} u_k(z)$ is uniformly convergent, we need to prove the following. Given any compact set K contained in Ω, and any $\varepsilon>0$, there exists $N>0$, for which the following statement holds true:

$$\text{for every } n\geq N, p\geq 0 \text{ and } z\in K, \text{ we have } \left|\sum_{k=n}^{n+p} u_k(z)\right|<\varepsilon.$$

(22.3) Power series and their radius of convergence.– A power series is an infinite series of the form $\sum_{k=0}^{\infty} a_k z^k$, where $a_0, a_1, \ldots \in \mathbb{C}$. That is, in the notation of the previous paragraph, each $u_k(z)=a_k z^k$. For instance, $\sum_{k=0}^{\infty} kz^k = 0+z+2z^2+3z^3+\cdots$ is a power series.

The following theorem is due to Abel [1].

Theorem. *Let $\sum_{k=0}^{\infty} a_k z^k$ be a power series. Then, there exists $R\in\mathbb{R}_{\geq 0}$ (R could be infinite) such that:*

1. *For every $z\in\mathbb{C}$ such that $|z|<R$, the series (of numbers) $\sum_{k=0}^{\infty} a_k z^k$ converges.*
2. *For every $z\in\mathbb{C}$ such that $|z|>R$, the series $\sum_{k=0}^{\infty} a_k z^k$ diverges.*

Assume that $R\neq 0$. Then the convergence of $\sum_{k=0}^{\infty} a_k z^k$ on the open disc $D(0;R)=\{z\in\mathbb{C} : |z|<R\}$ is uniform.

The non–negative real number R appearing in the statement of the theorem above is called *the radius of convergence* of the power series $\sum_{k=0}^{\infty} a_k z^k$.

The proof of this theorem is given in (22.8) below. It is optional, though it contains an important idea on how to find the radius of convergence, which I will highlight in the next

[1] Neils Henrik Abel (1802-1829)

paragraph.

(22.4) How to find the radius of convergence.– The radius of convergence of a power series $\sum_{k=0}^{\infty} a_k z^k$ is the unique non–negative real number R such that: for every non–negative, real number $r < R$, there exists some constant M (possibly depending on r) for which $|a_n|r^n < M$ for all $n \geq 0$.

Example. Let us consider the power series $1 + z + z^2 + z^3 + \ldots = \sum_{k=0}^{\infty} z^k$. Its radius of convergence is 1. If $0 \leq r < 1$, then with $M = 1$, we can be sure that $r^n < M$, for every $n \geq 0$. On the other hand, if $r > 1$, no such M could possibly exist as, in this case, $r^n \to \infty$ as $n \to \infty$.

d'Alembert's Ratio test to compute radius of convergence. One standard trick to compute the radius of convergence of $\sum_{k=0}^{\infty} a_k z^k$ is to compute the limit $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$. Let us assume this limit exists, and call it ℓ (could be 0 or ∞). Then $R = \frac{1}{\ell}$ ($R = \infty$ if $\ell = 0$, $R = 0$ if $\ell = \infty$).

(22.5) More examples.– Let us apply the ratio test to compute the radius of convergence of some useful power series.

(1) $\sum_{k=0}^{\infty} z^k$. Radius of convergence $= 1$ (see the example of the previous paragraph).

(2) $\sum_{k=0}^{\infty} \frac{z^k}{k!}$. In this case, the coefficients of the power series are $a_k = \frac{1}{k!}$, and

$$\frac{a_{n+1}}{a_n} = \frac{n!}{(n+1)!} = \frac{1}{n+1} \to 0 \text{ as } n \to \infty.$$

Thus, the radius of convergence $R = \infty$.

(3) $\sum_{k=0}^{\infty} kz^k$. $\frac{a_{n+1}}{a_n} = \frac{n+1}{n} = 1 + \frac{1}{n} \to 1$ as $n \to \infty$. Hence, $R = 1$.

(4) $\sum_{k=0}^{\infty} k!z^k$. $\frac{a_{n+1}}{a_n} = \frac{(n+1)!}{n!} = n + 1 \to \infty$ as $n \to \infty$. Hence, $R = 0$.

(22.6) Weierstrass' theorem on uniform convergence.– The following theorem was obtained by Weierstrass in his famous Berlin lectures around 1878.

Theorem. *Let $\Omega \subset \mathbb{C}$ be an open set, and let $\{f_n : \Omega \to \mathbb{C}\}_{n=0}^{\infty}$ be a sequence of functions. Assume $\{f_n\}_{n=0}^{\infty}$ converges uniformly, and let $f : \Omega \to \mathbb{C}$ be the limit.*

(1) *If each f_n is continuous, then so is f. In this case, for every piecewise–smooth path $\gamma : [a,b] \to \Omega$, we have*

$$\int_\gamma f(z)\,dz = \lim_{n\to\infty} \int_\gamma f_n(z)\,dz.$$

(2) *If each f_n is holomorphic, then so if f. In this case, $\{f_n'\}_{n=0}^\infty$ converges uniformly to f'.*

The point of the this theorem is that *once we have checked uniform convergence* the order of $\frac{d}{dz}$, $\int_\gamma$ and $\lim\limits_{n\to\infty}$ can be flipped:

$$\frac{d}{dz}\left(\lim_{n\to\infty} f_n(z)\right) = \lim_{n\to\infty} \frac{df_n}{dz},$$

$$\int_\gamma \left(\lim_{n\to\infty} f_n(z)\right)\,dz = \lim_{n\to\infty} \int_\gamma f_n(z)\,dz.$$

The proof of this theorem is given in the next paragraph, and is optional.

(22.7) Proof of Theorem (22.6).–[2] Recall that we are given an open set $\Omega \subset \mathbb{C}$ and a sequence of functions $\{f_n : \Omega \to \mathbb{C}\}_{n=0}^\infty$. We are assuming that this sequence converges uniformly to $f : \Omega \to \mathbb{C}$.

Proof of (1). Assume that each f_n is continuous. To show that f is continuous at a point $\alpha \in \Omega$, we have to prove the following: given any $\varepsilon > 0$, there exists some $\delta > 0$ for which the following statement holds true:

$$0 < |z-\alpha| < \delta \text{ implies that } |f(z) - f(\alpha)| < \varepsilon.$$

Let us first pick a positive real number $r > 0$ so that the closed disc $\overline{D(\alpha; r)} = \{z \in \mathbb{C} : |z-\alpha| \le r\}$ is contained in our domain Ω. This is a compact set contained in Ω, so by uniform convergence, we can find $N > 0$ such that:

$$|f_n(z) - f(z)| < \frac{\varepsilon}{3} \text{ for every } n \ge N \text{ and } z \in \overline{D(\alpha; r)}.$$

As $f_N(z)$ is continuous at α, there must exist some $\widetilde{\delta} > 0$, that makes the following statement true:

$$0 < |z-\alpha| < \widetilde{\delta} \text{ implies } |f_N(z) - f_N(\alpha)| < \frac{\varepsilon}{3}.$$

Now, the δ we were looking for, can be taken as: $\delta < \mathrm{Min}\{\widetilde{\delta}, r\}$. With this choice, whenever $0 < |z-\alpha| < \delta$, we have:

$$\begin{aligned}
|f(z) - f(\alpha)| &= |(f(z) - f_N(z)) + (f_N(z) - f_N(\alpha)) + (f_N(\alpha) - f(\alpha))| \\
&< |f(z) - f_N(z)| + |f_N(z) - f_N(\alpha)| + |f_N(\alpha) - f(\alpha))| \\
&< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.
\end{aligned}$$

Having established that the continuity of each f_n implies that of f, let us now show that $\int_\gamma f_n(z)\,dz \to \int_\gamma f(z)\,dz$ as $n \to \infty$, where $\gamma : [a,b] \to \Omega$ is a piecewise smooth path.

[2]Optional

Take the compact set $K = \{\gamma(t) : a \le t \le b\} \subset \Omega$ (this is just the image of γ inside Ω). Let $L = \text{length}(\gamma)$.

Now, let $\varepsilon > 0$ be given to us. We have to find $N > 0$ so that

$$\left| \int_\gamma f_n(z)\,dz - \int_\gamma f(z)\,dz \right| < \varepsilon \text{ for every } n \ge N.$$

By uniform continuity (applied to the compact set K), there does exist some $N > 0$, for which we have:

$$|f_n(z) - f(z)| < \frac{\varepsilon}{L} \text{ for every } n \ge N \text{ and } z \in K.$$

This N works for us, by the important inequality (Lecture 12, §12.7, page 9):

$$\left| \int_\gamma (f_n(z) - f(z))\,dz \right| < \frac{\varepsilon}{L} L = \varepsilon.$$

Proof of (2). Now we assume that each f_n is holomorphic. We are going to prove that f is also holomorphic. That is, for $w \in \Omega$, the limit $\lim_{h\to 0} \dfrac{f(w+h) - f(w)}{h}$ exists. This argument uses Cauchy's integral formula (Lecture 16, §16.3, page 4), and the idea behind the proof of "once differentiable always differentiable" (Lecture 17, page 4).

So, let us keep $w \in \Omega$ fixed. Pick $r > 0$ so that the closed disc $\overline{D(w;r)}$ is still in Ω and let C be the counterclockwise circle of radius r centered at w.

$$C : [0, 2\pi] \to \Omega \text{ given by } C(\theta) = w + re^{\mathbf{i}\theta}\ .$$

In the calculations below, I am assuming that $|h| < r$.

$$\begin{aligned}
\frac{1}{h}(f(w+h) - f(w)) &= \frac{1}{h}\lim_{n\to\infty}(f_n(w+h) - f_n(w)) \\
&= \frac{1}{h}\lim_{n\to\infty}\frac{1}{2\pi\mathbf{i}}\int_C \left(\frac{f_n(z)}{z-(w+h)} - \frac{f_n(z)}{z-w} \right) dz = \frac{1}{2\pi\mathbf{i}}\lim_{n\to\infty}\int_C \frac{f_n(z)}{(z-w-h)(z-w)}\,dz
\end{aligned}$$

This is because of Cauchy's integral formula applied to $f_n(z)$ which was assumed to be holomorphic. Now we can flip the role of $\int_C$ and $\lim_{n\to\infty}$, as we have already shown it to be legitimate in the proof of (1) above. Thus, we obtain:

$$\frac{1}{h}(f(w+h) - f(w)) = \frac{1}{2\pi\mathbf{i}}\int_C \frac{f(z)}{(z-w-h)(z-w)}\,dz.$$

Now, the limit as $h \to 0$, of the right–hand side exists, and is equal to $\dfrac{1}{2\pi\mathbf{i}}\displaystyle\int_C \frac{f(z)}{(z-w)^2}\,dz$ (see Lecture 17, page 4, where this is shown in detail). Hence, we can conclude that

$$f'(w) = \lim_{h\to 0}\frac{f(w+h) - f(w)}{h} \text{ exists and } = \frac{1}{2\pi\mathbf{i}}\int_C \frac{f(z)}{(z-w)^2}\,dz.$$

The same as true for each f_n by Cauchy's integral formula, which together with the proof of (1) above establishes that $\{f_n'(z)\}_{n=0}^\infty$ converges uniformly to $f'(z)$.

(22.8) Proof of Abel's theorem (22.3).–[3] Given a power series $\sum_{k=0}^{\infty} a_k z^k$, consider the following subset of $\mathbb{R}_{\geq 0}$:

$$I = \{r \in \mathbb{R}_{\geq 0} : \text{ there exists } M \text{ such that } |a_n| r^n < M, \text{ for every } n \geq 0\}$$

This is a non–empty set, since $0 \in I$. It is also an interval, meaning if $r \in I$ and $s < r$, then $s \in I$. *Let us define R to be the least upper bound of I.* Note that if $I = [0, \infty)$ then $R = \infty$. This is just to say that our interval I is of the form $[0, R)$ or $[0, R]$.

Proof of (2). Let $z \in \mathbb{C}$ be such that $|z| = r > R$. By our construction of R, this means the sequence $\{|a_n| r^n\}_{n=0}^{\infty}$ is unbounded. This implies that $\sum_{k=0}^{\infty} a_k z^k$ is divergent. The cleanest way to prove it is by contradiction. Let us assume $\sum_{k=0}^{\infty} a_k z^k$ is convergent. Take $\varepsilon = 1$ (for definiteness). Then, by definition of convergence, there must exist some $N > 0$ such that

$$|a_n z^n + a_{n+1} z^{n+1} + \cdots + a_{n+p} z^{n+p}| < 1, \text{ for every } n \geq N, p \geq 0.$$

In particular (with $p = 0$), this is saying that $|a_n z^n| = |a_n| r^n < 1$ for every $n \geq N$. Therefore, if we take

$$M = \text{Max}\{|a_0|, |a_1| r, \ldots, |a_{N-1}| r^{N-1}, 1\},$$

then $|a_n| r^n < M$ for every $n \geq 0$. But that means $r \in I$ and hence $r \leq R$. This is a contradiction to the initial assumption that $r > R$.

Proof of (1). Now, let us assume that $R > 0$ (if $R = 0$, there is nothing to prove in (1)). We will show the uniform convergence of $\sum_{k=0}^{\infty} a_k z^k$ on the open disc $D(0; R)$ (which clearly implies (1)).

So, let there be given a compact set $K \subset D(0; R)$. As K is bounded, there must be some number $r_1 < R$ such that $|z| \leq r_1$ for every $z \in K$. Pick any real number r_2 so that $r_1 < r_2 < R$. Again, by our construction of R, there is a constant M for which $|a_n| r_2^n < M$ for every $n \geq 0$.

With all this preparation, we are ready to prove the convergence of $\sum_{k=0}^{\infty} a_k z^k$ that is uniform for the compact set K. So, let $\varepsilon > 0$ be given to us. As $t = \frac{r_1}{r_2} < 1$, $\lim_{n \to \infty} t^n = 0$, so we can choose $N > 0$ such that $t^N \frac{M}{1-t} < \varepsilon$.

[3] Optional. The important bit is to see how the radius of convergence is defined.

Now, for every $n \geq N$, $p \geq 0$ and $z \in K$ we have:

$$\begin{aligned}
\left|\sum_{k=n}^{n+p} a_k z^k\right| &< \sum_{k=n}^{n+p} |a_k||z|^k \text{ (by triangle inequality)} \\
&< \sum_{k=n}^{n+p} |a_k| r_1^k < \sum_{k=n}^{n+p} \frac{M}{r_2^k} r_1^k \text{ (by definition of } M\text{: } |a_\ell| < \frac{M}{r_2^\ell} \text{ for every } \ell) \\
&= M t^n (1 + t + t^2 + \cdots + t^p) \text{ (recall } t = \frac{r_1}{r_2} < 1) \\
&< M t^N (1 + t + t^2 + \cdots) = \frac{M t^N}{1 - t} < \varepsilon.
\end{aligned}$$