# COMPLEX ANALYSIS: LECTURE 23

**(23.0) Review of the previous lecture.–** In Lecture 22, we considered a special type of series of functions, namely power series (see §22.3).

- A power series is a series of the form $\sum_{k=0}^{\infty} a_k z^k$ where $a_k \in \mathbb{C}$ for every $k \geq 0$.
- Associated to a power series $\sum_{k=0}^{\infty} a_k z^k$ is a real number $R \in \mathbb{R}_{\geq 0} \cup \{\infty\}$, called its radius of convergence. It is the unique number such that:
  - For every $0 \leq r < R$, we can find some constant $M$ for which $|a_n|r^n < M$ for every $n \geq 0$.
  - For every $r > R$, the sequence of numbers $\{|a_n|r^n\}_{n=0}^{\infty}$ is unbounded.

  In practice, we compute it using the ratio test:
  $$\frac{1}{R} = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|.$$
- Abel's theorem (§22.3) implies that within the open disc $D(0; R)$, the power series $\sum_{k=0}^{\infty} a_k z^k$ converges uniformly.
- Weierstrass' theorem (§22.6) on uniform convergence, in turn, implies that the limit of a uniformly convergent sequence of holomorphic functions is again holomorphic. Moreover, uniform convergence ensures that we can interchange the order of (a) taking the limit, and (b) taking derivative or integral over a piecewise smooth path.

**(23.1) Power series as holomorphic functions.–** Let us combine the two theorems stated above for power series. Let $\sum_{k=0}^{\infty} a_k z^k$ be a power series, and let $R$ be its radius of convergence. Assume that $R > 0$ (for $R = 0$, all the statements below are vacuous!).

(i) The sequence of partial sums, obtained from the power series is:

$$f_n(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n = \sum_{k=0}^{n} a_k z^k.$$

As each $f_n(z)$ is a polynomial, hence holomorphic, the power series $\sum_{k=0}^{\infty} a_k z^k$ is the (uniform) limit of holomorphic functions. By Weierstrass' theorem on uniform convergence, we conclude that:

$$\sum_{k=0}^{\infty} a_k z^k : D(0;R) \to \mathbb{C} \text{ is a holomorphic function.}$$

(ii) Now we can interchange the order of differentiation and limit to conclude that *power series can be differentiated term–wise*:

$$\boxed{\frac{d}{dz}\left(\sum_{k=0}^{\infty} a_k z^k\right) = \sum_{k=1}^{\infty} k a_k z^{k-1}}$$

(iii) Similarly, an anti–derivative of $\sum_{k=0}^{\infty} a_k z^k$ can be computed term–wise, and hence is: $\sum_{k=0}^{\infty} a_k \frac{z^{k+1}}{k+1}$.

*Exercise.* (Such problems are optional for our course, but do give it a try. Solution is given in (23.7) below). If $R$ is the radius of convergence of a power series $\sum_{k=0}^{\infty} a_k z^k$, then $R$ is also the radius of convergence of $\sum_{k=1}^{\infty} k a_k z^{k-1}$, and that of $\sum_{k=0}^{\infty} \frac{a_k}{k+1} z^{k+1}$.

**Example.** Consider the power series $A(z) = \sum_{k=0}^{\infty} z^k$. Its radius of convergence is 1. As a function, it is easy to see that $zA(z) = z + z^2 + \cdots = A(z) - 1$, therefore:

$$\sum_{k=0}^{\infty} z^k = \frac{1}{1-z} \text{ for } z \in D(0;1)$$

Note that the domain of the function $\frac{1}{1-z}$ is $\mathbb{C} \setminus \{1\}$. But the identity written above is only valid for $|z| < 1$.

**(23.2) Algebraic operations on power series.**– The usual addition and multiplication of polynomials carries over to power series. Let $A(z) = \sum_{k=0}^{\infty} a_k z^k$ and $B(z) = \sum_{\ell=0}^{\infty} b_\ell z^\ell$ be two power series, with radii of convergence $R_1$ and $R_2$ respectively. Then:

$$A(z) + B(z) = \sum_{k=0}^{\infty} (a_k + b_k) z^k \ ,$$

$$A(z) \cdot B(z) = \sum_{n=0}^{\infty} (a_0 b_n + a_1 b_{n-1} + \cdots a_n b_0) z^n = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} a_k b_{n-k}\right) z^n \ .$$

Both $A(z) + B(z)$ and $A(z) \cdot B(z)$ have their radii of convergence greater than, or equal to $\mathrm{Min}\{R_1, R_2\}$.

**(23.3) Power series centered at $\alpha \in \mathbb{C}$.–** Making the situation a little general, by a power series *centered at* $\alpha \in \mathbb{C}$, we mean a series of the form $\displaystyle\sum_{k=0}^{\infty} a_k(z-\alpha)^k$. A power series (with no mention of center, as before) is, by default, assumed to be centered at 0.

As before, a power series $\displaystyle\sum_{k=0}^{\infty} a_k(z-\alpha)^k$, centered at $\alpha$, with radius of convergence $R$, defines for us a holomorphic function on the open disc $D(\alpha; R) = \{z \in \mathbb{C} : |z-\alpha| < R\}$.

**(23.4) Taylor[1] series of a holomorphic function.–** Let $\Omega \subset \mathbb{C}$ be an open set, and let $f : \Omega \to \mathbb{C}$ be a holomorphic function. Let $\alpha \in \Omega$, and let $R$ be the largest positive real number such that the open disc $D(\alpha; R) \subset \Omega$.

**Theorem.** *We have a power series, centered at $\alpha$, of radius of convergence at least $R$, which is equal to $f$ on $D(\alpha; R)$:*

$$\sum_{n=0}^{\infty} c_n(z-\alpha)^n = f(z) \text{ for all } z \in D(\alpha; R).$$

*The coefficients $c_n$ ($n \geq 0$) are given by Cauchy's integral formula. Namely, let $C_r$ be the counterclockwise circle, centered at $\alpha$, of radius $r < R$. Then:*

$$\boxed{c_n = \frac{f^{(n)}(\alpha)}{n!} = \frac{1}{2\pi\mathbf{i}} \int_{C_r} \frac{f(z)}{(z-\alpha)^{n+1}}\, dz}$$

Proof. [2] Define $c_n = \dfrac{f^{(n)}(\alpha)}{n!}$. We will start by showing that the radius of convergence of the power series $\displaystyle\sum_{n=0}^{\infty} c_n(z-\alpha)^n$ is at least $R$.

Let us take $0 \leq r < R$. We need to come up with a constant $M$ so that $|c_n|r^n < M$ for every $n \geq 0$. So, let $C_r$ be the counterclockwise circle as in the statement of the theorem above. Choose $M$ to be larger than the absolute maximum of $|f(z)|$ on $C_r$. By Cauchy's integral formula, we have:

$$c_n = \frac{f^{(n)}(\alpha)}{n!} = \frac{1}{2\pi\mathbf{i}} \int_{C_r} \frac{f(z)}{(z-\alpha)^{n+1}}\, dz.$$

Now, we can use the important inequality to say:

$$|c_n| < \frac{1}{2\pi}\frac{M}{r^{n+1}} 2\pi r = \frac{M}{r^n}.$$

Thus, $|c_n|r^n < M$ for every $n \geq 0$, which is what we wanted to show.

[1] Brook Taylor (1685-1731) obtained this formal series in his book *Methodus incrementorum* dated 1715.
[2] This proof is due to Cauchy *Cours d'Analyse* 1822.

Hence, the power series $F(z) = \sum_{n=0}^{\infty} c_n(z-\alpha)^n$ converges uniformly on $D(\alpha; R)$ and defines a holomorphic function $F : D(\alpha; R) \to \mathbb{C}$. Now we have to prove that $F(w) = f(w)$ for every $w \in D(\alpha; R)$. To see this, let us write $w = \alpha + h$, where $|h| < R$. Choose $r$ to be between $|h|$ and $R$: $|h| < r < R$. Then, again by Cauchy's integral formula:

$$f(\alpha+h) = \frac{1}{2\pi \mathbf{i}} \int_{C_r} \frac{f(z)}{z-\alpha-h}\, dz.$$

For $z$ on the circle $C_r$, we have $|z-\alpha| = r > |h|$. Therefore we can expand (see Example in §23.1 above):

$$\frac{1}{z-\alpha-h} = \frac{1}{z-\alpha} \cdot \frac{1}{1-\frac{h}{z-\alpha}} = \frac{1}{z-\alpha} \sum_{k=0}^{\infty} \left(\frac{h}{z-\alpha}\right)^k = \sum_{k=0}^{\infty} \frac{h^k}{(z-\alpha)^{k+1}}.$$

Substituting it back in the expression for $f(\alpha+h)$ above, and flipping the order of integral and sum (valid by uniform convergence), we get:

$$f(\alpha+h) = \sum_{k=0}^{\infty} h^k \frac{1}{2\pi \mathbf{i}} \int_{C_r} \frac{f(z)}{(z-\alpha)^{k+1}}\, dz = \sum_{k=0}^{\infty} \frac{f^{(k)}(\alpha)}{k!} h^k = F(\alpha+h).$$

This is exactly what we wanted to verify.

□

**(23.5) Examples.**– Let us compute the Taylor series expansions of some holomorphic functions.

(1) $f(z) = e^z$. Compute the Taylor series of $f$ cenetered at 0.
*Solution.* Since $f^{(n)}(z) = e^z$ for every $n \geq 0$ and $f(0) = 1$, we get:

$$e^z = 1 + \frac{z}{1!} + \frac{z^2}{2!} + \ldots = \sum_{k=0}^{\infty} \frac{z^k}{k!}\ .$$

The radius of convergence of the power series above is $\infty$ (see Example (2) of §22.5). This can also be viewed from the theorem above, since the domain of $e^z$ is $\mathbb{C}$. So, the number $R$ appearing at the beginning of the previous section can be chosen to be as large as we please.

(2) $f(z) = \sin(z)$. Again, we are going to compute the Taylor series centered at 0.
*Solution.* Note that $f'(z) = \cos(z)$, $f''(z) = -\sin(z)$. Continuing this way, we see that $f^{(2k)}(z) = f(z) = (-1)^k \sin(z)$ and $f^{(2k+1)}(z) = (-1)^k \cos(z)$. Using $\sin(0) = 0$ and $\cos(0) = 1$, we get:

$$\sin(z) = 0 + z - 0\frac{z^2}{2!} - 1\frac{z^3}{3!} + \ldots = \sum_{k=0}^{\infty} (-1)^k \frac{z^{2k+1}}{(2k+1)!}\ .$$

Again, the radius of convergence of the Taylor series is $\infty$, because $\sin(z)$ is defined for all $z \in \mathbb{C}$.
*Alternate solution.* We can also use the definition $\sin(z) = \dfrac{e^{\mathbf{i}z} - e^{-\mathbf{i}z}}{2\mathbf{i}}$ and the Taylor series of $e^z$ computed above to get the same answer (you should try to do this by yourself).

(3) $f(z) = \ln(z)$, centered at 1 (see Lecture 9, §9.1, page 2).
*Solution.* $f(1) = 0$. $f'(z) = z^{-1}$, $f''(z) = -z^{-2}$, $f^{(3)}(z) = 2z^{-3}$ and continuing this way, we have:

$$f^{(n)}(z) = (-1)^{n-1}(n-1)!z^{-n} \text{ for every } n \geq 1.$$

Therefore, $f^{(n)}(1) = (-1)^{n-1}(n-1)!$. Hence, the Taylor series expansion of $\ln(z)$, centered at 1 is given by:

$$\ln(z) = \sum_{n=1}^{\infty}(-1)^{n-1}\frac{(n-1)!}{n!}(z-1)^n = \sum_{n=1}^{\infty}(-1)^{n-1}\frac{(z-1)^n}{n} .$$

The radius of convergence of this series is 1. Reason 1: the largest open we can have, centered at 1, on which $\ln(z)$ is defined, must have radius 1 (recall $\ln(z)$ is not defined for $z \in \mathbb{R}_{\leq 0}$). Reason 2: use ratio test to compute the radius of convergence of $\displaystyle\sum_{n=1}^{\infty}(-1)^{n-1}\frac{z^n}{n}$.

(4) $f(z) = \dfrac{1}{(z-1)(z-2)}$. Taylor series centered at 0.
*Solution.* Computing repeated derivatives is going to be difficult. Fortunately, for rational functions, we can use partial fractions to do the computation as follows.

$$\frac{1}{(z-1)(z-2)} = \frac{1}{z-2} - \frac{1}{z-1}.$$

Now, we have (see Example in §23.1 above)

$$\frac{1}{z-1} = -\frac{1}{1-z} = -\sum_{k=0}^{\infty} z^k ,$$

$$\frac{1}{z-2} = -\frac{1}{2}\cdot\frac{1}{1-(z/2)} = -\frac{1}{2}\sum_{k=0}^{\infty}(z/2)^k = -\sum_{k=0}^{\infty}\frac{1}{2^{k+1}}z^k .$$

So, the Taylor series expansion is given by:

$$\frac{1}{(z-1)(z-2)} = \sum_{n=0}^{\infty}\left(1 - \frac{1}{2^{n+1}}\right)z^n .$$

(Convince yourself that the radius of convergence is 1)

**(23.6) Power series centered at $\infty$.**– A power series centered at $\infty$ is just a power series in $z^{-1}$. Namely, a series of the form $\displaystyle\sum_{k=0}^{\infty} c_k z^{-k}$. There is no significant difference here: if $R$ is the radius of convergence of the series $\displaystyle\sum_{k=0}^{\infty} c_k w^k$, then $\displaystyle\sum_{k=0}^{\infty} c_k z^{-k}$ converges uniformly within the open set:

$$D(\infty; R) = \left\{z \in \mathbb{C} : |z| > \frac{1}{R}\right\} \subset \mathbb{C}.$$

**(23.7) Solution to Exercise from (23.1).**–[3] Let $R$ be the radius of convergence of the power series $\displaystyle\sum_{k=0}^{\infty} a_k z^k$. Recall that, this means:

- For every $0 \leq r < R$, there is a constant (depending on $r$) $M$ such that $|a_n|r^n < M$, for every $n \geq 0$.
- For every $r > R$, the sequence of numbers $\{|a_n|r^n\}_{n=0}^{\infty}$ is unbounded.

We want to show that $R$ is the radius of convergence of $\displaystyle\sum_{k=1}^{\infty} k a_k z^{k-1}$. That means, given $0 \leq r < R$, we must find a constant $M$ for which $n|a_n|r^{n-1} < M$ for every $n \geq 1$.

So, pick an $r_1$ between $r$ and $R$: $r < r_1 < R$, and let $M_1$ be the constant so that $|a_n|r_1^n < M_1$ for every $n \geq 0$. Then, we have:

$$n|a_n|r^{n-1} < n\frac{M_1}{r_1^n}r^{n-1} = \frac{M_1}{r} \cdot n\left(\frac{r}{r_1}\right)^n .$$

Let $t = \frac{r}{r_1} < 1$. Then $\lim\limits_{n\to\infty} nt^n = 0$ (verify this!). So, (taking $\varepsilon = 1$ in the definition of the limit) we can choose $N > 0$ so that $nt^n < 1$ for every $n \geq N$. Then we get:

$$n|a_n|r^{n-1} < \frac{M_1}{r} \text{ for every } n \geq N.$$

So, the $M$ we are looking for, can be taken to be:

$$M > \mathrm{Max}\left\{\{k|a_k|r^{k-1}\}_{k=1}^{N-1}, \frac{M_1}{r}\right\}.$$

This proves that the radius of convergence of $\displaystyle\sum_{k=1}^{\infty} k a_k z^{k-1}$, say $R_1$, is greater than, or equal to $R$ (that is, $R_1 \geq R$).

Prove that the radius of convergence of an anti–derivative of a power series is also greater than or equal to that of the power series itself (second part of the exercise). As $\displaystyle\sum_{k=0}^{\infty} a_k z^k$ is an anti–derivative of $\displaystyle\sum_{k=1}^{\infty} k a_k z^{k-1}$, we will obtain $R \geq R_1$, hence they must be equal.

[3] Optional