COMPLEX ANALYSIS: LECTURE 24

Special thanks to Prof. Maria Angelica Cueto for drawing the pictures for these notes.

(24.0) Review.– In the last two lectures, we started the study of power series. To summarize:

$$\begin{tikzcd} \boxed{\text{Power series centered at } \alpha} \arrow[d, leftrightsquigarrow] \\ \boxed{\text{Holomorphic functions with } \alpha \text{ in their domain}} \end{tikzcd}$$

- From top to bottom: given a power series centered at $\alpha \in \mathbb{C}$, $A(z) = \displaystyle\sum_{k=0}^{\infty} a_k(z-\alpha)^k$, with radius of convergence R, Abel's theorem (§22.2) + Weierstrass' theorem (§22.6) imply that $A : D(\alpha; R) \to \mathbb{C}$ is a holomorphic function. (recall: $D(\alpha; R) = \{z \in \mathbb{C} : |z-\alpha| < R\}$).

- From bottom to top: given a holomorphic function defined on an open set Ω, $f : \Omega \to \mathbb{C}$, and $\alpha \in \Omega$, Taylor series expansion (§23.4) gives us:
$$f(z) = \sum_{k=0}^{\infty} \frac{f^{(k)}(\alpha)}{k!}(z-\alpha)^k \ .$$
This identity is valid on an open disc $D(\alpha; R)$, where $R \in \mathbb{R}_{>0}$ is the largest number we can have so that $D(\alpha; R) \subset \Omega$.

(24.1) Isolated singularity.– Now we are going to consider the situation when the point of interest α is **not** in the domain of our function. The set up is as follows (see the figure below).

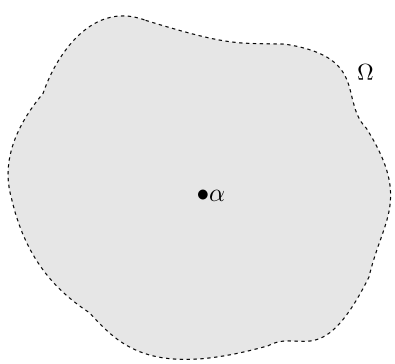

FIGURE 1. The domain of our function f is $\Omega \setminus \{\alpha\}$.

- $\Omega \subset \mathbb{C}$ is an open set, and $\alpha \in \Omega$.

- $f : \Omega \setminus \{\alpha\} \to \mathbb{C}$ is a holomorphic function. Meaning, f is defined on an open set around α, but (possibly) not at α.

In this case, we say α is an *isolated singularity* of f.

(24.2) Examples.– Let us look at a few examples of functions with an isolated, and non–isolated singularity.

(1) $f(z) = z^{-n}$, $n \in \mathbb{Z}_{\geq 1}$. Its domain is $\mathbb{C} \setminus \{0\}$ and 0 is its isolated singularity.

(2) $f(z) = \dfrac{1}{\sin(z)} = \mathrm{cosec}(z)$. Since $\sin(z) = 0 \iff z = n\pi$ for some $n \in \mathbb{Z}$, the domain of $\mathrm{cosec}(z)$ is $\mathbb{C} \setminus \{0, \pm\pi, \pm 2\pi, \ldots\}$. For every integer n, the point $n\pi$ is an isolated singularity of $\mathrm{cosec}(z)$.

(3) $f(z) = e^{\frac{1}{z}}$. Again 0 is the only (hence isolated) singularity of this function.

(4) Let $f(z) = \dfrac{\sin(z)}{z}$ be defined on $\mathbb{C} \setminus \{0\}$. 0 is (technically speaking) an isolated singularity of this function. The point of this example is: it may be possible to define $f(z)$ at the missing point - later, we will call such points *removable singularity* (see §24.5 below).

(5) $f(z) = \mathrm{cosec}\left(\dfrac{1}{z}\right)$. Comparing with the example number (2) above, the singularities of this function are $\left\{\frac{1}{n\pi} : n \in \mathbb{Z}_{\neq 0}\right\}$ and 0. In this case, 0 is *not* isolated!

(24.3) Laurent series.– [1] Let us return back to our general set up from §24.1. Thus f is a holomorphic function on $\Omega \setminus \{\alpha\}$ (see Figure 1 above). Let $R \in \mathbb{R}_{>0}$ be such that the open disc $D(\alpha; R) \subset \Omega$.

Let r be any positive real number less than R: $0 < r < R$ (see Figure 2 below).

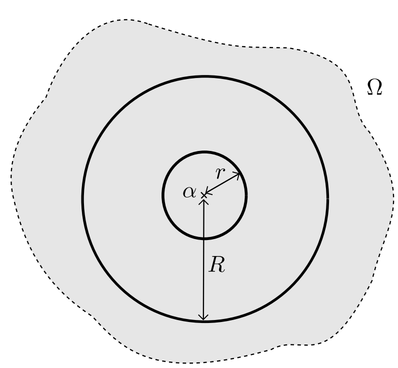

FIGURE 2. Circles centered at α of radii $r < R$ respectively.

[1] Pierre Alphonse Laurent (1813-1854). Laurent published his theorem is 1843 (Comptes Rendus XVII, 348-349). The same result appeared in an earlier paper of Weierstrass (1841), but unfortunately, Weierstrass' paper was not published until 1894 (posthumously).

Let $\mathrm{Ann}(\alpha; r < R)$ denote the *annular region* between the two circles. Thus,

$$\boxed{\mathrm{Ann}(\alpha; r < R) = \{z \in \mathbb{C} : r < |z - \alpha| < R\}}$$

Theorem. *For every $w \in \mathrm{Ann}(\alpha; r < R)$ we have the following:*

$$f(w) = \sum_{k=0}^{\infty} c_k (w - \alpha)^k + \sum_{\ell=1}^{\infty} \frac{d_\ell}{(w - \alpha)^\ell} \ .$$

The coefficients $\{c_k\}_{k=0}^{\infty}$ and $\{d_\ell\}_{\ell=1}^{\infty}$ are given by:

$$\boxed{c_k = \frac{1}{2\pi \mathbf{i}} \int_{C_R} \frac{f(z)}{(z - \alpha)^{k+1}}\, dz} \ \textit{and}\ \boxed{d_\ell = \frac{1}{2\pi \mathbf{i}} \int_{C_r} (z - \alpha)^{\ell - 1} f(z)\, dz}$$

Here, C_r and C_R are counterclockwise circles, centered at α of radii r and R respectively.

A proof of this theorem is given in §24.8 below. It is almost identical to the one for Taylor's theorem given in Lecture 23, §23.4.

(24.4) Remarks.–

(1) The series appearing in the statement of the theorem above is called *Laurent series* of f centered at α. The identity

$$f(z) = \sum_{k=0}^{\infty} c_k (z - \alpha)^k + \sum_{\ell=1}^{\infty} \frac{d_\ell}{(z - \alpha)^\ell}$$

is known as the *Laurent series expansion* of f around α.

(2) In the theorem above, we can take r to be as small as we want, but not zero! Similarly, R can be taken as large so as to still have $D(\alpha; R) \subset \Omega$.

(3) Note that we can no longer say $c_k = \dfrac{f^{(k)}(\alpha)}{k!}$ (as we did in Taylor's theorem from Lecture 23, §23.4). This is simply because f is not defined at α.

(4) Implicit in the statement of the theorem above is the assertion: both series

$$F^+(z) = \sum_{k=0}^{\infty} c_k (z - \alpha)^k \qquad \text{and} \qquad F^-(z) = \sum_{\ell=1}^{\infty} \frac{d_\ell}{(z - \alpha)^\ell}$$

converge uniformly in the open set $\mathrm{Ann}(\alpha; r < R)$. This assertion is proved exactly as the uniform convergence of Taylor series, namely, by estimating the modulus of the coefficients, using our important inequality (see the proof of Theorem 23.4, Lecture 23 page 3). So, I will not rewrite it here.

(24.5) Classification of isolated singularities.– Using Laurent's theorem (24.3) above, we have three possibilities for an isolated singularity α:

Removable singularity. If $d_\ell = 0$ for every $\ell \geq 1$, then the Laurent series becomes Taylor series. In this case, α is not a genuine singularity: we can extend f by setting $f(\alpha) = c_0$ and have a holomorphic function $\Omega \to \mathbb{C}$. We say α *is a removable singularity.*

Pole of order $n \in \mathbb{Z}_{\geq 1}$. The second possibility is that $d_\ell' s$ eventually become zero. More precisely, there is a positive integer n such that:

$$d_{n+1} = d_{n+2} = \cdots = 0 \qquad \text{and} \qquad d_n \neq 0.$$

In this case, we say α *is a pole of order* n. (By the terminology that is established here: *a pole of order* 0 *is a removable singularity.*)

Essential singularity. The only remaining case is when $d_\ell' s$ never stop being non–zero. To write it precisely: for every $n \geq 1$, there is some $m > n$ such that $d_m \neq 0$. In this case, we say α *is an essential singularity.*

(24.6) How to determine the order of a pole.– Let us assume that α is a pole of order n, for a given function $f : \Omega \setminus \{\alpha\} \to \mathbb{C}$. By definition, this means that the Laurent series expansion of f around α has the form, with $d_n \neq 0$:

$$f(z) = \frac{d_n}{(z-\alpha)^n} + \frac{d_{n-1}}{(z-\alpha)^{n-1}} + \cdots + \frac{d_1}{z-\alpha} + \sum_{k=0}^{\infty} c_k (z-\alpha)^k \ .$$

Thus, $g(z) = (z-\alpha)^n f(z)$ is defined at α (or, in more technical words: α is a removable singularity of $g(z)$):

$$g(\alpha) = \lim_{z \to \alpha} (z-\alpha)^n f(z) = d_n \neq 0.$$

This gives us a very practical way to determine the order of the pole of f at α:

$$\boxed{\alpha \text{ is a pole of order } n \in \mathbb{Z}_{\geq 1} \text{ of } f \iff \lim_{z \to \alpha} (z-\alpha)^n f(z) \text{ exists and } \neq 0}$$

The argument I wrote above only proves $\Rightarrow$. The converse is also true, and is proved by applying Taylor's theorem to $g(z)$ which gives Laurent series of $f(z) = \dfrac{g(z)}{(z-\alpha)^n}$. I will write it in more detail in the next lecture.

(24.7) Examples.– Let us see what happens in the examples of §24.2 above.

(1) $f(z) = \dfrac{1}{z^n}$ where $n \in \mathbb{Z}_{\geq 1}$. Clearly 0 is a pole of order n here.

(2) $f(z) = \operatorname{cosec}(z)$. Let us look at the singularity at $z = 0$. By l'Hôpital rule:

$$\lim_{z \to 0} z \operatorname{cosec}(z) = 1 (\neq 0).$$

So, $f(z)$ has a pole of order 1 at $z = 0$. Similarly, it has a pole of order 1 at each $z = n\pi$, $n \in \mathbb{Z}$.

(3) $f(z) = e^{\frac{1}{z}}$. Using the Taylor series expansion of the exponential function, we have:

$$e^{\frac{1}{z}} = \sum_{\ell=0}^{\infty} \frac{1}{\ell!} \frac{1}{z^\ell}.$$

This is the third possibility listed in §24.5 above. Hence, 0 is an essential singularity of $e^{\frac{1}{z}}$.

(4) $f(z) = \dfrac{\sin(z)}{z}$, $\alpha = 0$. We know the Taylor series expansion of $\sin(z)$ near 0 (see Lecture 23, §23.5, example (2)):

$$\sin(z) = z - \frac{z^3}{3!} + \frac{z^5}{5!} + \cdots$$

Therefore, we get the Laurent series expansion

$$\frac{\sin(z)}{z} = 1 - \frac{z^2}{3!} + \frac{z^4}{5!} + \cdots$$

Since there are no terms with negative exponent of z, 0 is a removable singularity of $\dfrac{\sin(z)}{z}$.

(5) $f(z) = \operatorname{cosec}\left(\dfrac{1}{z}\right)$, near 0. This example was that of a *non–isolated singularity.* We haven't discussed it yet - but non–isolated singularities are always essential.

(24.8) Proof of Laurent's Theorem (24.3).– As mentioned earlier, the proof of this theorem is entirely analoguous to that of Lecture 23, Theorem 23.4. Recall the set up from §24.3 above (see Figure 2).

Let $w \in \mathrm{Ann}(\alpha; r < R)$. Consider the counterclockwise oriented concentric circles C_r and C_R (see Figure 3 below - the inner circle is given clockwise orientation in the figure, so it is written as $-C_r$).

By Cauchy's integral formula (see Lecture 16, §16.1 - for the principle of contour deformation; and §16.3 - for the integral formula) we have:

$$f(w) = \frac{1}{2\pi\mathbf{i}} \int_{C_R - C_r} \frac{f(z)}{z - w}\, dz.$$

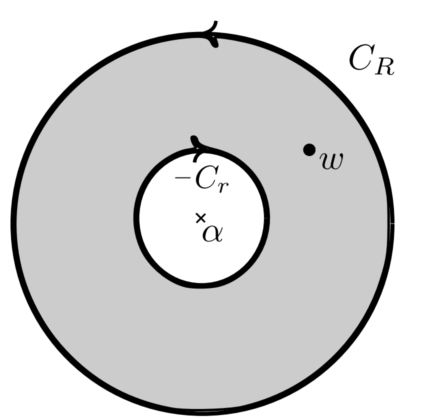

FIGURE 3. Contours C_R and $-C_r$

(Convince yourself of this. *Hint: Pick a point P on C_R and a point Q on C_r. Draw a straightline, joining P to Q - call it μ. Consider the closed path γ: start at P, go around C_R, followed by μ, C_r and $-\mu$ in that order. $\gamma = C_R + \mu - C_r - \mu$ then has w in its interior, and $f(z)$ is holomorphic within it. Now Cauchy's integral formula applies.*)

Now that we have:

$$f(z) = \frac{1}{2\pi\mathbf{i}} \int_{C_R} \frac{f(z)}{z-w}\, dz - \frac{1}{2\pi\mathbf{i}} \int_{C_r} \frac{f(z)}{z-w}\, dz \ ,$$

we can compute each term individually, as follows.

(1) For z on C_R, $|z-\alpha| = R > |w-\alpha|$. Therefore, we have the geometric series expansion (compare with the argument on page 4 of Lecture 23):

$$\frac{1}{z-w} = \frac{1}{(z-\alpha)-(w-\alpha)} = \frac{1}{z-\alpha} \cdot \frac{1}{1-\frac{w-\alpha}{z-\alpha}} = \frac{1}{z-\alpha} \sum_{k=0}^{\infty} \frac{(w-\alpha)^k}{(z-\alpha)^k}$$

Thus, we obtain (order of $\int$ and $\sum_{k=0}^{\infty}$ can be interchanged, by the uniform convergence of geometric series [2], and Weierstrass' theorem §22.6):

$$\int_{C_R} \frac{f(z)}{z-w}\, dz = \sum_{k=0}^{\infty} \left(\int_{C_R} \frac{f(z)}{(z-\alpha)^{k+1}}\, dz \right) (w-\alpha)^k \ .$$

(2) Similarly, for z on C_r, $|z-\alpha| = r < |w-\alpha|$ and we have:

$$\frac{1}{z-w} = -\frac{1}{(w-\alpha)-(z-\alpha)} = -\frac{1}{w-\alpha} \sum_{\ell=0}^{\infty} \frac{(z-\alpha)^\ell}{(w-\alpha)^\ell} \ .$$

This gives:

$$\int_{C_r} \frac{f(z)}{z-w}\, dz = -\sum_{\ell=0}^{\infty} \left(\int_{C_r} (z-\alpha)^\ell f(z)\, dz \right) \frac{1}{(w-\alpha)^{\ell+1}} \ .$$

Combining the two calculations, implies:

$$f(w) = \sum_{k=0}^{\infty} c_k (w-\alpha)^k + \sum_{\ell=1}^{\infty} \frac{d_\ell}{(w-\alpha)^\ell} \ ,$$

where:

$$c_k = \frac{1}{2\pi\mathbf{i}} \int_{C_R} \frac{f(z)}{(z-\alpha)^{k+1}}\, dz \qquad \text{and} \qquad d_\ell = \frac{1}{2\pi\mathbf{i}} \int_{C_r} (z-\alpha)^{\ell-1} f(z)\, dz \ .$$

The theorem is proved.

[2] The series $\sum_{k=0}^{\infty} z^k$ is called the geometric series.