COMPLEX ANALYSIS: LECTURE 25

(25.0) What is in these notes.–

(1) I will define order of vanishing of a holomorphic function at a point in §25.2. This definition is built upon the Taylor series expansion from Lecture 23, §23.4 (reviewed in §25.1). We will learn that if $f(z_0) = 0$, then either f is identically zero in a neighbourhood of z_0, or there is a neighbourhood of z_0 in which f vanishes precisey at z_0 (Lemma 25.2 below).

(2) The previous result is used to prove the *identity theorem* in §25.3. It says that if two holomorphic functions (defined on an open and connected set) agree on any converging sequence, they have to agree on the entire domain.

(3) Next, we will review the notion of a pole of order N in §25.4. The situation is parallel to that of a zero of order N - as explained in §25.5.

(4) In §25.6: the residue of a function at an isolated singularity is defined. Two methods of computing the residue are explained in §25.7. Four examples featuring these methods are worked out in §25.8.

(25.1) Taylor series - review.– Recall Theorem (23.4) from Lecture 23: given a holomorphic function $f : \Omega \to \mathbb{C}$, defined on an open set $\Omega \subset \mathbb{C}$, and a point $z_0 \in \Omega$, we have (Taylor series expansion):

$$f(z) = \sum_{n=0}^{\infty} c_n (z - z_0)^n \text{ for every } z \in D(z_0; R).$$

- The coefficients $\{c_n\}_{n=0}^{\infty}$ are given by: $\boxed{c_n = \dfrac{f^{(n)}(z_0)}{n!}}$
- $R \in \mathbb{R}_{>0}$ can be taken to be as large as we please, under the condition that $D(z_0; R) \subset \Omega$.
  (Recall: $D(z_0; R) = \{z \in \mathbb{C} : |z - z_0| < R\}$.)

Remark. In Lecture 17, we saw the first name change, from $\mathbb{C}$–differentiable to holomorphic to honour the "once differentiable, always differentiable" property of such functions. The existence of Taylor series expansion is another very important property of holomorphic functions, for which holomorphic functions are given yet another name: *analytic* [1].

As I mentioned in Lecture 17, for functions of a real variable, all these names mean different things. There exist functions of a real variable which are differentiable to all orders, but

[1] I like holomorphic though, and will keep using it - it has Greek origins: holo=entire, morph=form.

do not admit Taylor series expansion. For instance: $f(x) = \begin{cases} e^{-\frac{1}{x^2}} & x \neq 0 \\ 0 & x = 0 \end{cases}$ near $x = 0$.

(25.2) Order of vanishing.– Let us take the Taylor series expansion of f near z_0 from the previous paragraph: $f(z) = \displaystyle\sum_{n=0}^{\infty} c_n(z-z_0)^n$, for every $z \in D(z_0; R)$. Assume that $f(z_0) = c_0 = 0$. There are two possibilities:

(Case 1) $c_n = 0$ for every $n \geq 0$. That is to say: $f(z) = 0$ for every $z \in D(z_0; R)$. This case is often written as $f \equiv 0$ (read: f is identically zero) on $D(z_0; R)$.

(Case 2) There is a number $N \in \mathbb{Z}_{\geq 1}$ such that

$$c_0 = c_1 = \cdots = c_{N-1} = 0 \qquad \text{and} \qquad c_N \neq 0.$$

This number N is called the *order of vanishing of f at z_0*. Another sentence to the same meaning is *z_0 is a zero of f, of order N*.

Lemma. *Let $f : \Omega \to \mathbb{C}$ be a holomorphic function, defined on an open set Ω. Let $z_0 \in \Omega$ be such that $f(z_0) = 0$. Then, the following are the only two possibilities:*

(1) *Either $f \equiv 0$ on $D(z_0; R)$,*

(2) *Or there exists $0 < r \leq R$ such that $f(z) \neq 0$ for every z in the punctured disc $D^{\times}(z_0; r)$.*
(Recall: $D^{\times}(z_0; r) = D(z_0; r) \setminus \{z_0\} = \{z \in \mathbb{C} : 0 < |z - z_0| < r\}$.)

PROOF. By the argument preceding the statement of the lemma, if f is not identically zero on $D(z_0; R)$, then there is a number $N \in \mathbb{Z}_{\geq 1}$ so that:

$$f(z) = \sum_{n=N}^{\infty} c_n(z-z_0)^n = (z-z_0)^N \sum_{k=0}^{\infty} c_{N+k}(z-z_0)^k \text{ for every } z \in D(z_0; R).$$

Let us call $g(z) = \displaystyle\sum_{k=0}^{\infty} c_{N+k}(z-z_0)^k$. Since the radius of convergence is still R, $g(z)$ defines a holomorphic function $D(z_0; R) \to \mathbb{C}$ such that $g(z_0) = c_N \neq 0$.

Since g is continuous, there must be some disc (say of radius r, $0 < r \leq R$) such that $g(z) \neq 0$ for every $z \in D(z_0; r)$. This is precisely the statement of the second case given above:

$$\text{for every } z \in D^{\times}(z_0; r) \text{ we have } f(z) = (z-z_0)^N g(z) \neq 0.$$

$\square$

(25.3) Identity Theorem.– The following result was advertised in Lecture 20, page 5.

Theorem. *Let $\Omega \subset \mathbb{C}$ be an open and connected set. Let $f_1, f_2 : \Omega \to \mathbb{C}$ be two holomorphic functions. Assume that there exists a sequence of points $\{w_n\}_{n=0}^{\infty} \subset \Omega$ which converges to a* *point $w \in \Omega$.*

If $f_1(w_n) = f_2(w_n)$ for every $n \geq 0$, then $f_1(z) = f_2(z)$ for every $z \in \Omega$ (equivalent way of writing this is: $f_1 \equiv f_2$ on Ω - read as: f_1 is identically equal to f_2 on Ω).

Proof. [2] First of all, let $F = f_1 - f_2 : \Omega \to \mathbb{C}$. This turns the hypothesis to $F(w_n) = 0$ for every $n \geq 0$, and we have to show $F(z) = 0$ for every $z \in \Omega$.

It is clear that $F(w) = 0$, since $w = \lim_{n\to\infty} w_n$, and F is continuous. Moreover, F cannot fall into the second possibility listed in Lemma (25.2) above, since $w_n' s$ converge to w. So, $F \equiv 0$ on some disc around w.

Now, let $z \in \Omega$ be an arbitrary point. Since Ω is connected, we can find a path $\gamma : [0,1] \to \Omega$ such that $\gamma(0) = w$ and $\gamma(1) = z$ (see Figure 1 below).

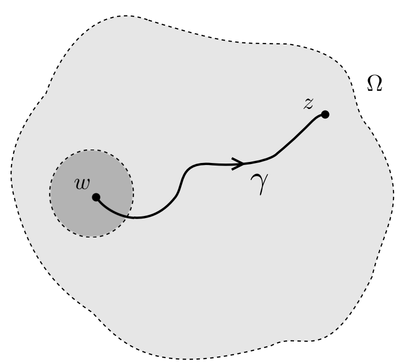

Figure 1. Path γ joining w to z. $F \equiv 0$ on the darkly shaded disc around w.

Let $S \subset [0,1]$ be defined by:

$$S = \{t \in [0,1] : F \equiv 0 \text{ on } \gamma([0,t])\}.$$

As $0 \in S$, this set is non–empty. Moreover, it is an interval: if $t \in S$ and $s < t$, then $s \in S$. Therefore, it must be of the form $S = [0,T]$ for some $T \in [0,1]$. We want to show that $T = 1$, which will imply that $F(z) = F(\gamma(1)) = 0$.

If $T < 1$, then using Lemma (25.2) again, with $z_0 = \gamma(T)$, and using the fact that $F(\gamma(t)) = 0$ for every $0 \leq t < T$, we conclude that we are still in the first possibility stated in the lemma: there must be some disc around $\gamma(T)$ on which $F \equiv 0$. Thus for some $\varepsilon > 0$, $T+\varepsilon \in S$ contradicting the fact that $S = [0,T]$. Hence T has to be 1, and the proof is finished.

$\square$

(25.4) Pole of order N - review.– Let us recall the notion of an *isolated singularity* from Lecture 24. The word *singularity* of a function f, simply refers to the points where f may

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[2] Optional. The argument given here is inspired by Lebasgue's proof of Heine–Borel theorem, see Optional Reading A, §A.7.

not be defined. A singularity $\alpha \in \mathbb{C}$ of f is *isolated*, if there is a disc $D(\alpha; R)$ of positive radius $R > 0$, such that f is defined on the punctured disc $D^\times(\alpha; R)$ (see Figure 2, page 2 of Lecture 24).

In case of an isolated singularity, we have the Laurent series expansion (see Theorem 24.3, page 3 of Lecture 24):

$$f(z) = \sum_{\ell=1}^{\infty} \frac{d_\ell}{(z-\alpha)^\ell} + \sum_{k=0}^{\infty} c_k (z-\alpha)^k \text{ for every } z \in D^\times(\alpha; R).$$

We say that f has a pole of order $N \in \mathbb{Z}_{\geq 1}$ at α, (or, α is an order N pole of f) if:

$$d_n = 0 \text{ for every } n > N, \qquad \text{and} \qquad d_N \neq 0.$$

That is, the Laurent series expansion of f near α is:

$$f(z) = \frac{d_N}{(z-\alpha)^N} + \frac{d_{N-1}}{(z-\alpha)^{N-1}} + \cdots + \frac{d_1}{z-\alpha} + \sum_{k=0}^{\infty} c_k (z-\alpha)^k \ .$$

Let us prove the claim made in Lecture 24, §24.6, page 4:

$$\boxed{\alpha \text{ is a pole of order } N \text{ of } f \iff \lim_{z\to\alpha} (z-\alpha)^N f(z) \text{ exists, and } \neq 0}$$

Proof. The forward implication is clear from the Laurent series, since in this case $d_N = \lim_{z\to\alpha} (z-\alpha)^N f(z)$ is non–zero.

For the converse, let us define

$$g(z) = \begin{cases} (z-\alpha)^{N+1} f(z), & z \neq \alpha \\ 0, & z = \alpha \end{cases}$$

Thus, g is defined on $\text{Domain}(f) \cup \{\alpha\}$. To see that g is holomorphic at α, we have (since $g(\alpha) = 0$):

$$\lim_{z\to\alpha} \frac{g(z) - g(\alpha)}{z-\alpha} = \lim_{z\to\alpha} \frac{g(z)}{z-\alpha} = \lim_{z\to\alpha} (z-\alpha)^N f(z),$$

which exists and is non–zero, by assumption.

g is clearly holomorphic at points in the domain of f. Consider its Taylor series expansion around α (Theorem (23.4) of Lecture 23): $g(z) = \sum_{k=1}^{\infty} a_k (z-\alpha)^k$. It starts at 1 because $g(\alpha) = 0$. Moreover, $a_1 \neq 0$ since $g'(\alpha) \neq 0$ by the calculation given above.

Therefore, around α, f has the following (Laurent series) expansion:

$$f(z) = \sum_{k=1}^{N} \frac{a_k}{(z-\alpha)^{N+1-k}} + \sum_{k=N+1}^{\infty} a_k (z-\alpha)^{k-N-1}.$$

Its leading term $a_1 \neq 0$. Hence, f has a pole of order N at α.

$\square$

(25.5) Pole of order N - summary.– To summarize previous paragraph, f has a pole of order N at α if, and only if we can write $f(z) = \dfrac{\varphi(z)}{(z-\alpha)^N}$, where φ is holomorphic around α and $\varphi(\alpha) \neq 0$.

More precisely, the following is the Laurent series expansion of f near α:

$$f(z) = \frac{d_N}{(z-\alpha)^N} + \frac{d_{N-1}}{(z-\alpha)^{N-1}} + \cdots + \frac{d_1}{z-\alpha} + \sum_{k=0}^{\infty} c_k(z-\alpha)^k \text{ for every } z \in D^{\times}(\alpha; R) \ ,$$

if, and only if the following is the Taylor series expansion of $\varphi(z) = (z-\alpha)^N f(z)$ near α:

$$\varphi(z) = d_N + d_{N-1}(z-\alpha) + \cdots + d_1(z-\alpha)^{N-1} + \sum_{k=0}^{\infty} c_k(z-\alpha)^{N+k} \text{ for every } z \in D(\alpha; R).$$

(25.6) Residues.– Let us continue with the set up that α is an isolated singularity (of any kind - removable, pole or essential) of f. Thus, there is a positive real number R such that f is defined on the punctured disc $D^{\times}(\alpha; R)$. Let C_r be the counterclockwise circle, centered at α of radius $r < R$.

Definition. The residue of f at α, denoted by $\operatorname*{Res}_{z=\alpha}(f(z))$, is defined as:

$$\boxed{\operatorname*{Res}_{z=\alpha}(f(z)) = \frac{1}{2\pi\mathbf{i}} \int_{C_r} f(z)\, dz}$$

By the principle of contour deformation (see Lecture 16, §16.1), C_r in the definition above can be replaced by any positively oriented contour γ such that:

- $\alpha \in \text{Interior}(\gamma)$.
- $\text{Interior}(\gamma) \setminus \{\alpha\}$ and γ itself are in the domain of f.

(25.7) How to compute the residue.– The very first method for computing the residue is Cauchy's integral formula (Lecture 17, §17.1). Let us review it:

The set up is: $\Omega \subset \mathbb{C}$ is an open set. $F : \Omega \to \mathbb{C}$ is a holomorphic function. $\gamma : [a,b] \to \Omega$ is a positively oriented contour such that $\text{Interior}(\gamma) \subset \Omega$. $w \in \text{Interior}(\gamma)$. Then, for every $n \in \mathbb{Z}_{\geq 0}$:

$$\boxed{\frac{1}{2\pi\mathbf{i}} \int_{\gamma} \frac{F(z)}{(z-w)^{n+1}}\, dz = \frac{F^{(n)}(w)}{n!}}$$

Thus, if $f(z)$ has a pole of order N at α, so that we can write (as in the previous paragraph) $f(z) = \dfrac{\varphi(z)}{(z-\alpha)^N}$, where $\varphi(\alpha) \neq 0$ (this is important!), then:

$$\boxed{\operatorname*{Res}_{z=\alpha}(f(z)) = \frac{\varphi^{(N-1)}(\alpha)}{(N-1)!}}$$

The second way to compute the residue is to write down the Laurent series expansion of f near α:

$$f(z) = \sum_{\ell=1}^{\infty} \frac{d_\ell}{(z-\alpha)^\ell} + \sum_{k=0}^{\infty} c_k (z-\alpha)^k \text{ for every } z \in D^\times(\alpha; R).$$

The convergence of the (two) series on the right–hand side is uniform, which allows us to interchange the roles of $\int$ and $\sum$ (Weierstrass' theorem 22.6, Lecture 22). The following computation is easy to prove (using Cauchy's integral formula, or antiderivative theorem). Here C_r is the counterclockwise circle from the definition of residue above.

$$\text{For } m \in \mathbb{Z}, \text{ we have } \frac{1}{2\pi \mathbf{i}} \int_{C_r} (z-\alpha)^m \, dz = \begin{cases} 1 & \text{if } m = -1 \\ 0 & \text{otherwise} \end{cases}$$

Hence, $\boxed{\operatorname*{Res}_{z=\alpha}(f(z)) = d_1}$, the coefficient of $(z-\alpha)^{-1}$ in the Laurent series expansion of f near α.

(25.8) Examples.– Let us compute the reside for some functions, using methods explained in the previous paragraph.

(1) $f(z) = \dfrac{1}{z(z-2)^5}$. Let C_0 be the counterclockwise circle around 0 of radius 1/2 (anything less than 2).

$$\operatorname*{Res}_{z=0}(f(z)) = \frac{1}{2\pi \mathbf{i}} \int_{C_0} \frac{1}{z(z-2)^5} \, dz = \left[\frac{1}{(z-2)^5} \right]_{\text{Set } z=0} = -\frac{1}{32}.$$

Similarly, let C_2 be a contour around 2 that does not contain 0 on it, or in its interior. Then,

$$\operatorname*{Res}_{z=2}(f(z)) = \frac{1}{2\pi \mathbf{i}} \int_{C_2} \frac{1}{z(z-2)^5} \, dz = \frac{1}{4!} \left[\frac{d^4}{dz^4} \left(z^{-1} \right) \right]_{\text{Set } =2} = \frac{1}{32}.$$

(Can you give an *a priori* justification why these numbers had to add up to zero?)

(2) $f(z) = e^{\frac{1}{z}}$. Compute $\operatorname*{Res}_{z=0}(f(z))$.
The singularity is essential, so Cauchy's integral formula is not going to help us. We have to write the Laurent series expansion: $e^{\frac{1}{z}} = \displaystyle\sum_{k=0}^{\infty} \frac{1}{k!} z^{-k}$.
Thus $\operatorname*{Res}_{z=0}(f(z)) = 1 =$ the coefficient of z^{-1}.

(3) $f(z) = \dfrac{1}{z(e^z - 1)}$. Note that, by l'hôpital rule: $\displaystyle\lim_{z\to 0} \frac{z}{e^z - 1} = 1$. Therefore, writing $f(z) = \dfrac{1}{z^2} \dfrac{z}{e^z - 1}$, we see that $f(z)$ has a pole of order 2 at $z = 0$. In the notation of §25.7 above, $\varphi(z) = \dfrac{z}{e^z - 1}$.

In Homework 6, Problem 13, you computed the first few terms of $\dfrac{z}{e^z - 1}$:

$$\frac{z}{e^z - 1} = 1 - \frac{z}{2} + \cdots$$

Hence, $\operatorname*{Res}_{z=0}(f(z)) =$ coefficient of z^{-1} in $\dfrac{1}{z^2} \cdot \left(1 - \dfrac{z}{2} + \cdots\right) = -\dfrac{1}{2}$.

(4) $f(z) = \dfrac{\ln(z)}{z^2+1}$. Here $\ln : \mathbb{C} \setminus \mathbb{R}_{\leq 0} \to \mathbb{C}$ was defined in Lecture 9, page 2. Recall:

$$\ln(z) = \ln(|z|) + \mathbf{i}\arg(z), \text{ where } -\pi < \arg(z) < \pi.$$

Take C to be a counterclockwise oriented circle around $\mathbf{i}$ of radius $r < 1$ (so as to avoid $\mathbb{R}_{\leq 0}$ as well as $-\mathbf{i}$).

$$\operatorname*{Res}_{z=\mathbf{i}}\left(\frac{\ln(z)}{(z-\mathbf{i})(z+\mathbf{i})}\right) = \frac{1}{2\pi\mathbf{i}}\int_C \frac{\ln(z)}{(z-\mathbf{i})(z+\mathbf{i})}\,dz = \left[\frac{\ln(z)}{z+\mathbf{i}}\right]_{\text{Set } z=\mathbf{i}} = \frac{\ln(\mathbf{i})}{2\mathbf{i}} = \frac{\pi}{4}\ .$$