COMPLEX ANALYSIS: LECTURE 26

(26.0) .– Recall that we defined $\operatorname{Res}_{z=\alpha}(f(z))$ for an isolated singularity $\alpha \in \mathbb{C}$ of a function f. Thus, f is defined, and holomorphic on the punctured disc of some radius $R \in \mathbb{R}_{>0}$, $f : D^\times(\alpha; R) \to \mathbb{C}$. (Recall: $D^\times(\alpha; R) = \{z \in \mathbb{C} : 0 < |z - \alpha| < R\}$.)

Let γ be a positively oriented contour such that (i) $\alpha \in \text{Interior}(\gamma)$, and (ii) $\text{Interior}(\gamma)\setminus\{\alpha\}$ and γ itself are in the domain of f. Then,

$$\boxed{\operatorname{Res}_{z=\alpha}(f(z)) = \frac{1}{2\pi\mathbf{i}}\int_\gamma f(z)\,dz}$$

We saw in Lecture 25, §25.7, that if α is a pole of f, we can use Cauchy's integral formula to compute the residue. In any case (whether α is a pole or essential singularity), we have: $\operatorname{Res}_{z=\alpha}(f(z)) =$ coefficient of $(z-\alpha)^{-1}$ in the Laurent series expansion of f near α.

(26.1) The point at ∞.– There is nothing special here. The behaviour of a function $f(z)$ near $z = \infty$ is investigated by performing the change of variables $z = \dfrac{1}{w}$ and studying the point $w = 0$ of $f\left(\dfrac{1}{w}\right)$. In more detail:

(1) $z = \infty$ is a singularity of $f(z)$ if $w = 0$ is a singularity of $f\left(\dfrac{1}{w}\right)$.

(2) Singularity at $z = \infty$ of $f(z)$ is isolated if $w = 0$ is an isolated singularity of $f\left(\dfrac{1}{w}\right)$.

(3) $z = \infty$ is a removable singularity, a pole, or an essential singularity of $f(z)$ if so is $w = 0$ of $f\left(\dfrac{1}{w}\right)$.

Examples. (1) e^z has an essential singularity at ∞. (2) $f(z) = \dfrac{z+1}{z^2-3}$ has a removable singularity at ∞. (3) Let $P(z)$ be a polynomial of degree n. Then $P(z)$ has a pole of order n at ∞ (see §26.7 for a converse to this statement).

The following exercise is given here for you to get used to the statements written above.

Exercise. Let $f : \Omega \to \mathbb{C}$ be a holomorphic function, defined on an open set Ω. Prove that ∞ is an isolated singularity of f if, and only if there is a number $R \in \mathbb{R}_{>0}$ such that $\{z \in \mathbb{C} : |z| > R\} \subset \Omega$. In this case, verify the following assertions:

(1) $f(z)$ has a removable singularity at ∞ if, and only if $\lim_{z\to\infty} f(z)$ exists.

(2) $f(z)$ has a pole of order N at ∞ if, and only if $\lim_{z\to\infty} z^{-N}f(z)$ exists, and is non–zero.

(26.2) Residue at ∞.– Again, assume that ∞ is an isolated singularity of f. Meaning, there is a number $R \in \mathbb{R}_{>0}$ so that f is defined (and holomorphic, as always) on $\{z \in \mathbb{C} : |z| > R\}$. Thus, all the singularities of f are contained in the closed disc: $\overline{D(0;R)} = \{z \in \mathbb{C} : |z| \leq R\}$.

Let γ be a *clockwise oriented* contour, large enough so as to have all the singularities of f in the interior of γ. For instance, pick a number $\rho > R$ and let $\gamma = -C_\rho$ be the *clockwise oriented* circle of radius ρ, centered at 0. We define:

$$\boxed{\operatorname*{Res}_{z=\infty}(f(z)) = \frac{1}{2\pi\mathbf{i}}\int_\gamma f(z)\,dz}$$

Remark. In the definition of $\operatorname*{Res}_{z=\infty}(f(z))$, the contour used is *negatively* oriented. The reason is - as one travels along γ, the point at ∞ should be to the left. I always use the notation C for *counterclockwise oriented* circles, that is why there is a minus sign in $\gamma = -C_\rho$.

The relation between residue of $f(z)$ at ∞, and the residue of $f(w^{-1})$ at 0 is obtained as follows. Upon the change of variables $w = z^{-1}$, we get $dw = -z^{-2}dz = -w^2dz$. Thus $dz = -\dfrac{dw}{w^2}$.

Moreover (easy check) as z moves over $-C_\rho$ (clockwise circle of radius ρ, centered at 0), $w = z^{-1}$ moves over $C_{\rho^{-1}}$ (counterclockwise circle of radius ρ^{-1}, centered at 0). This gives:

$$\int_{-C_\rho} f(z)dz = \int_{C_{\rho^{-1}}} f\left(w^{-1}\right)\frac{(-1)dw}{w^2}.$$

Hence we arrive at the following identity.

$$\boxed{\operatorname*{Res}_{z=\infty}(f(z)) = -\operatorname*{Res}_{w=0}\left(w^{-2}f\left(w^{-1}\right)\right)}$$

Example. Let us take a rational function $f(z) = \dfrac{3z^2+1}{(z-1)(2z-1)^2}$. It only has poles at $z = 1$ and $z = 1/2$. Let us compute its residue at ∞ in two different ways.
(i) Directly from the definition. Take $\rho = 2$ (anything bigger than 1 will suffice), and let C_ρ be the counterclockwise circle of radius ρ, centered at 0. I will leave the following calculation to you (it actually follows easily from a result that we proved in Lecture 19, §19.5, page 7).

$$\frac{1}{2\pi\mathbf{i}}\int_{C_\rho} f(z)\,dz = \frac{3}{4}.$$

Hence $\operatorname*{Res}_{z=\infty}(f(z)) = -\dfrac{3}{4}$.

(ii) Using the change of variables $z = w^{-1}$.

$$f(w^{-1}) = \frac{3w^{-2}+1}{(w^{-1}-1)(2w^{-1}-1)^2} = \frac{w(3+w^2)}{(1-w)(2-w)^2}.$$

Now we use $\operatorname*{Res}_{z=\infty}(f(z)) = -\operatorname*{Res}_{w=0}\left(w^{-2}f(w^{-1})\right)$ to carry out the calculation. Let C be a small enough counterclockwise circle centered at 0 (small enough = radius < 1).

$$\operatorname*{Res}_{z=\infty}(f(z)) = -\frac{1}{2\pi\mathbf{i}}\int_C \frac{3+w^2}{w(1-w)(2-w)^2}\,dw = -\left[\frac{3+w^2}{(1-w)(2-w)^2}\right]_{\text{Set } w=0} = -\frac{3}{4}.$$

(26.3) A word on the non–isolated case.– Recall that an example of a non–isolated singularity was given in Lecture 24, §24.2: $f(z) = \operatorname{cosec}\left(z^{-1}\right)$. The set of singularities is $\left\{\frac{1}{n\pi} : n \in \mathbb{Z}_{\neq 0}\right\} \cup \{0\}$, and 0 is not isolated. Meaning, no matter how small radius $r > 0$ we pick, the disc $D(0;r)$ will contain a singularity other than 0 (in fact, infinitely many).

In general, $\alpha \in \mathbb{C}$ is a non–isolated singularity of f, if there exists a sequence $\{\alpha_n\}_{n=0}^{\infty}$ of complex numbers such that:

- Each α_n is a *non–removable* singularity of f (it could be a pole, or an essential singularity).
- $\lim_{n\to\infty} \alpha_n = \alpha$.

Note that in this case α cannot be removable, or a pole - if it were there would exist some positive real number $r \in \mathbb{R}_{>0}$ such that f is defined on the punctured disc $D^{\times}(\alpha; r)$ (see, for instance, Lecture 25, §25.4). The existence of such r will contradict the fact that $\alpha_n \to \alpha$ as $n \to \infty$.

Hence, *non–isolated singularities can only be essential.*

(26.4) Cauchy's residue theorem.– Let f be a holomorphic function, and γ be a positively oriented contour. Assume that there are finitely many points $\alpha_1, \alpha_2, \ldots, \alpha_n \in \text{Interior}(\gamma)$ such that $\text{Interior}(\gamma) \setminus \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ and γ itself are in the domain of f.

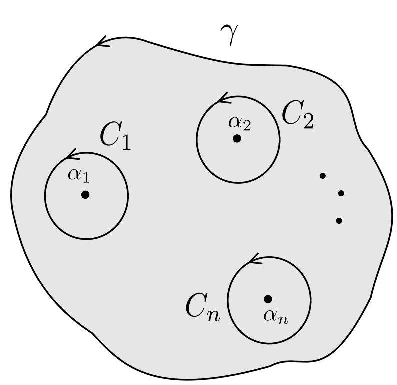

FIGURE 1. f is a holomorphic function defined on an open set, which contains γ and $\text{Interior}(\gamma) \setminus \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$.

Theorem.

$$\boxed{\frac{1}{2\pi \mathbf{i}}\int_\gamma f(z)\,dz = \sum_{j=1}^{n} \operatorname*{Res}_{z=\alpha_j}(f(z))}$$

Proof. The proof of this theorem is based on the principle of contour deformation (Lecture 16, §16.1). Namely, let C_j be a small enough, counterclockwise oriented circle, around α_j such that, for $k \neq j$, α_k is not on, or in the interior of C_j (see Figure 1 above). By the principle of contour deformation:

$$\frac{1}{2\pi \mathbf{i}}\int_\gamma f(z)\,dz = \sum_{j=1}^{n} \frac{1}{2\pi \mathbf{i}}\int_{C_j} f(z)\,dz = \sum_{j=1}^{n} \operatorname*{Res}_{z=\alpha_j}(f(z))\,.$$

The last equality is by definition of the residue. □

Example. Let C be the counterclockwise oriented circle of radius 2, centered at 0. Compute $\displaystyle\int_C \frac{e^{\pi z}}{z(z^2+1)}\,dz$.

$$\begin{aligned}
\int_C \frac{e^{\pi z}}{z(z^2+1)}\,dz &= 2\pi \mathbf{i}\left(\operatorname*{Res}_{z=0}\left(\frac{e^{\pi z}}{z(z^2+1)}\right) + \operatorname*{Res}_{z=\mathbf{i}}\left(\frac{e^{\pi z}}{z(z^2+1)}\right) + \operatorname*{Res}_{z=-\mathbf{i}}\left(\frac{e^{\pi z}}{z(z^2+1)}\right)\right)\\
&= 2\pi \mathbf{i}\left(\left[\frac{e^{\pi z}}{z^2+1}\right]_{\text{Set } z=0} + \left[\frac{e^{\pi z}}{z(z+\mathbf{i})}\right]_{\text{Set } z=\mathbf{i}} + \left[\frac{e^{\pi z}}{z(z-\mathbf{i})}\right]_{\text{Set } z=-\mathbf{i}}\right)\\
&= 2\pi \mathbf{i}\left(1 + \frac{-1}{\mathbf{i}(2\mathbf{i})} + \frac{-1}{-\mathbf{i}(-2\mathbf{i})}\right)\\
&= 4\pi \mathbf{i}.
\end{aligned}$$

(26.5) Meromorphic functions.– A function f is called *meromorphic* [1] on an open set $\Omega \subset \mathbb{C}$, if there exists a subset $A \subset \Omega$ such that:

- $f : \Omega \setminus A \to \mathbb{C}$ is holomorphic.
- Every $\alpha \in A$ is either a removable singularity, or a pole of f (that is, f is not allowed to have essential singularities in Ω).

Example. Every holomorphic function is also meromorphic, with $A = \emptyset$. $\operatorname{cosec}(z)$ and $\cot(z)$ are meromorphic functions on $\mathbb{C}$, with $A = \{n\pi : n \in \mathbb{Z}\}$. $e^{1/z}$ is not meromorphic on $\mathbb{C}$ (though it is holomorphic on $\mathbb{C} \setminus \{0\}$, it has an essential singularity at 0).

A rational function, for instance $f(z) = \dfrac{z^3+z+3}{z^5-1}$, is meromorphic on $\mathbb{C}$ (in the example, $f(z)$ has poles at the fifth roots of unity).

[1] Greek: holo = whole/entire, mero = part. This is to highlight that f is only defined on a *part* of Ω.

The following notation is sometimes used for meromorphic functions: $f : \Omega \dashrightarrow \mathbb{C}$ to indicate that f may not be defined on a subset $A \subset \Omega$. I am going to assume that every point of A is a pole (by including removable singularities in the domain of f). A pole is always isolated, so the set A itself has to be *discrete* (meaning, for every $\alpha \in A$, there exists $r \in \mathbb{R}_{>0}$ such that the punctured disc around α of radius r does not intersect A: $D^\times(\alpha; R) \cap A = \emptyset$). This leads to the following important property of meromorphic functions.

Proposition. *Let $K \subset \mathbb{C}$ be a closed and bounded (in other words, compact) set, such that $K \subset \Omega$. Then $K \cap A$ is finite.*

Proof. This is an application of Bolzano–Weierstrass theorem[2] : an infinite collection of points in a compact set always have a cluster point. More precisely, if $\{\alpha_n\}_{n=0}^{\infty}$ is contained in a compact set K, then there exists $\alpha \in K$ with the property that for every $r > 0$, there are infinitely many $\alpha_n' s$ with $|\alpha - \alpha_n| < r$.

Assume, for the sake of obtaining a contradiction, that $K \cap A$ is infinite. By Bolzano–Weierstrass theorem, we will have a point $\alpha \in K \subset \Omega$, such that for every $r > 0$, $D^\times(\alpha; r) \cap A \neq \emptyset$. This α is then a *non–isolated*, hence essential singularity of f (see §26.3 above). This contradicts the assumption that f is meromorphic (f is not allowed to have essential singularities in Ω). □

(26.6) Application 1: sum of residues.– Let $f : \mathbb{C} \dashrightarrow \mathbb{C}$ be a meromorphic function. Assume that ∞ is an isolated singularity of f.

This means that there is a number $R \in \mathbb{R}_{>0}$ so that f is defined and holomorphic on the open set $\{z \in \mathbb{C} : |z| > R\}$ (see §26.1 above). Hence, all the singularities (which are necessarily poles since f is meromorphic) are contained in the compact set $K = \{z \in \mathbb{C} : |z| \leq R\}$. Therefore, by Proposition 26.5 above, there are only finitely many of them, say $\alpha_1, \alpha_2, \ldots, \alpha_n$.

Cauchy's residue theorem (§26.4), and the definition of $\operatorname*{Res}_{z=\infty}(f(z))$ (§26.2) imply:

$$\boxed{\sum_{j=1}^{n} \operatorname*{Res}_{z=\alpha_j}(f(z)) + \operatorname*{Res}_{z=\infty}(f(z)) = 0}$$

(26.7) Application 2: entire functions with a pole at ∞.– The following argument is a mild generalization of the one we saw in the proof of Liouville's theorem (Lecture 18, §18.2, page 3).

Let $f : \mathbb{C} \to \mathbb{C}$ be a holomorphic function defined on the entire complex plane (such functions are called *entire functions*). Assume that f has a pole of order N at ∞.

Claim. f is a polynomial of degree N.

[2]A proof of Bolzano–Weierstrass theorem (over $\mathbb{R}$) is given in Optional Reading A, §A.2. I will leave it to you (if you are interested) to carry out the proof over $\mathbb{C}$ - hint: use the version over $\mathbb{R}$ for the real and imaginary parts.

Proof. Let us take the Taylor series expansion of f near 0: $f(z) = \sum_{k=0}^{\infty} a_k z^k$. As f is entire, the radius of convergence of this series is ∞, by Theorem (23.4) of Lecture 23.

As f has a pole of order N at ∞, by the exercise given in §26.1 above, $\lim_{z\to\infty} z^{-N} f(z)$ exists and is non–zero. In particular it implies, by definition of the limit, that there is a number $R \in \mathbb{R}_{>0}$, and a constant $M \in \mathbb{R}_{>0}$ so that

$$|z^{-N} f(z)| < M \text{ for every } z \in \mathbb{C} \text{ such that } |z| > R.$$

The coefficients of the Taylor series of $f(z)$ can be computed as follows. Take any $\rho > 0$, and let C_ρ be the counterclockwise circle of radius ρ centered at 0. Then:

$$a_k = \frac{1}{2\pi\mathbf{i}} \int_{C_\rho} \frac{f(z)}{z^{k+1}}\, dz.$$

(we can take $\rho > 0$ to be as large as we want, since f is defined on the entire complex plane).

Now, taking $\rho > R$, we can estimate $|a_k|$ for $k \geq N+1$, using the bound of $|z^{-N} f(z)|$ written above, and our important inequality (see Lecture 12, §12.7, page 9). For every $\ell \geq 0$:

$$|a_{N+1+\ell}| = \left| \frac{1}{2\pi\mathbf{i}} \int_{C_\rho} \frac{f(z)}{z^{N+\ell+2}}\, dz \right| < \frac{1}{2\pi} \frac{M}{\rho^{\ell+2}} 2\pi\rho = \frac{M}{\rho^{\ell+1}}$$

As $\dfrac{M}{\rho^{\ell+1}} \to 0$ as $\rho \to \infty$, we conclude that $|a_{N+1+\ell}| = 0$ for every $\ell \geq 0$. Hence, $f(z) = a_0 + a_1 z + \cdots + a_N z^N$ is a polynomial.

Note that $\lim_{z\to\infty} z^{-N} f(z) = a_N$ which was assumed to be non–zero. Hence degree of f is precisely N. □

(26.8) Meromorphic functions on $\mathbb{C}$ with a pole at ∞.–

Theorem. *Let us assume that we have a meromorphic function $f : \mathbb{C} \dashrightarrow \mathbb{C}$ which has a pole at ∞ (in particular, ∞ is an isolated singularity of f).*

Then, $f(z)$ is a rational function.

Proof. Using the argument given in §26.6 above, we know f will only have finitely many poles $\{\alpha_1, \alpha_2, \ldots, \alpha_n\} \subset \mathbb{C}$. For any $1 \leq j \leq n$, we have the Laurent series expansion of f near α_j (remember: it is a pole, say of order N_j):

$$f(z) = \frac{d^{(j)}_{N_j}}{(z-\alpha_j)^{N_j}} + \cdots + \frac{d^{(j)}_1}{z-\alpha_j} + \sum_{k=0}^{\infty} c^{(j)}_k (z-\alpha_j)^k .$$

Let us define:

$$R_j(z) = \frac{d_{N_j}^{(j)}}{(z-\alpha_j)^{N_j}} + \cdots + \frac{d_1^{(j)}}{z-\alpha_j}.$$

Then, each $R_j(z)$ is a rational function, with only pole at α_j. Moreoever, $\lim_{z\to\infty} R_j(z) = 0$.

Let us define $g(z) = f(z) - \sum_{j=1}^{n} R_j(z)$.

Claim. $g(z)$ is holomorphic on the entire complex plane.

Let us assume this claim to be true and proceed with the proof of the theorem. Note that if $f(z)$ has a pole of order N at ∞, then so does $g(z)$. This is because $\lim_{z\to\infty} z^{-N} R_j(z) = 0$, for every j, and therefore $\lim_{z\to\infty} z^{-N} g(z) = \lim_{N\to\infty} z^{-N} f(z)$. Hence $g(z)$ is a polynomial of degree N, by §26.7 above. We can now conclude that $f(z) = g(z) + \sum_{j=1}^{n} R_j(z)$ is a rational function, being a sum of a polynomial and finite number of rational functions.

Proof of the claim. For $w \in \mathbb{C}$, $w \notin \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$, it is clear that g is holomorphic at w (since f and $R_j's$ are). So, let us consider the situation near $w = \alpha_j$ ($j \in \{1, 2, \ldots, n\}$). Each $R_\ell(z)$, with $\ell \neq j$, is holomorphic at α_j, since the only singularity of R_ℓ is α_ℓ. Moreover, near α_j, we have the following expansion $f(z) - R_j(z) = \sum_{k=0}^{\infty} c_k^{(j)} (z-\alpha_j)^k$, which means α_j is a removable singularity of $f(z) - R_j(z)$. Hence we conclude that

$$g(z) = f(z) - R_j(z) - \sum_{\substack{\ell=1 \\ \ell\neq j}}^{n} R_\ell(z) \text{ is holomorphic at } \alpha_j \ .$$

So $g(z)$ has no singularities in the entire complex plane. The claim follows. □