COMPLEX ANALYSIS: LECTURE 27

(27.0) Residue theorem - review.– In these notes we are going to use Cauchy's residue theorem to compute some real integrals. Let us recall the statement of this theorem. We are given a holomorphic function f (on some open set - domain of f), a counterclockwise oriented contour γ, and a finite collection of points $\alpha_1, \alpha_2, \ldots, \alpha_n \in \text{Interior}(\gamma)$. This data is supposed to satisfy the following assumption (see Figure 1 below).

The set $\text{Interior}(\gamma) \setminus \{\alpha_1, \ldots, \alpha_n\}$, and the contour γ itself are both in the domain of f.

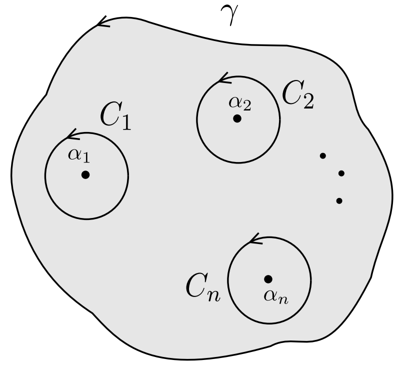

FIGURE 1. f is a holomorphic function defined on an open set, which contains γ and $\text{Interior}(\gamma) \setminus \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$.

The residue theorem is just a combination of the principle of contour deformation and the definition of residue at an isolated singularity. It says:

$$\boxed{\int_\gamma f(z)\, dz = 2\pi \mathbf{i} \sum_{j=1}^{n} \operatorname*{Res}_{z=\alpha_j}(f(z))}$$

(27.1) Applications to real integrals.– There are four types of real integrals which we are going to try to compute with the help of the residue theorem. These notes contain the first two classes of examples (other two will be in Lecture 28).

Class I. $\displaystyle\int_0^{2\pi} R(\cos(\theta), \sin(\theta))\, d\theta$. Here the integrand is usually a rational expression involving $\cos(\theta)$ and $\sin(\theta)$ which remains finite within the limits of integration.

Method. We make the change of variables: $z = e^{\mathbf{i}\theta}$, so that $dz = \mathbf{i}e^{\mathbf{i}\theta} d\theta$. This transforms our (real, definite) integral to contour integration, over C : the counterclockwise oriented circle of radius 1, centered at 0.

$$\cos(\theta) \mapsto \frac{z+z^{-1}}{2}, \qquad \sin(\theta) \mapsto \frac{z-z^{-1}}{2\mathbf{i}}, \qquad d\theta \mapsto \frac{1}{\mathbf{i}z} dz\ .$$

$$\boxed{\int_0^{2\pi} R(\cos(\theta), \sin(\theta))\, d\theta = \int_C R\left(\frac{z+z^{-1}}{2}, \frac{z-z^{-1}}{2\mathbf{i}}\right) \frac{1}{\mathbf{i}z}\, dz}$$

The contour integral on the right–hand side of the equation above can be computed using the residue theorem (see Examples given in §27.2, 27.3 below).

Class II. P. V. $\int_{-\infty}^{\infty} Q(x)\, dx$, defined as $\lim_{R\to\infty} \int_{-R}^{R} Q(x)\, dx$.

Remark. The prefix P. V. stands for *principal value.* Without this prefix, the integral $\int_{-\infty}^{\infty} Q(x)\, dx$ is defined as:

$$\int_{-\infty}^{\infty} Q(x)\, dx = \lim_{\substack{R_1\to\infty \\ R_2\to\infty}} \int_{-R_2}^{R_1} Q(x)\, dx.$$

The two are not the same, in general. For instance P. V. $\int_{-\infty}^{\infty} x\, dx = 0$, while $\int_{-\infty}^{\infty} x\, dx$ does not exist [1].

The method given below computes the principal value integral only. In most of our examples, $Q(x)$ will be an even function. For such functions P. V. $\int_{-\infty}^{\infty} Q(x)\, dx = \int_{-\infty}^{\infty} Q(x)\, dx$.

Method. For problems of this type, we let C_R be the counterclockwise contour consisting of two smooth parts: $\mu_R =$ the straightline joining $-R$ to R, and $\gamma_R =$ semicircle in the upper half of the complex plane, joining R to $-R$ (see Figure 2 below). Then:

$$\int_{-R}^{R} Q(x)\, dx = \int_{\mu_R} Q(z)\, dz = \int_{C_R} Q(z)\, dz - \int_{\gamma_R} Q(z)\, dz.$$

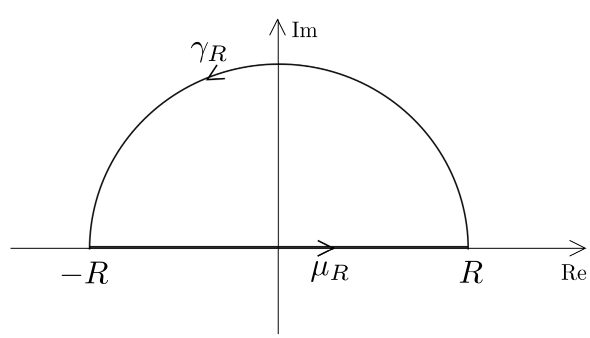

FIGURE 2. Contour C_R consisting of two parts: straight line segment μ_R and a semicircle γ_R

[1] $\int_{-\infty}^{\infty} Q(x)\, dx$ exists if, and only if both $\int_{-\infty}^{0} Q(x)\, dx$ and $\int_{0}^{\infty} Q(x)\, dx$ exist *individually*, in which case: $\int_{-\infty}^{\infty} Q(x)\, dx = \int_{-\infty}^{0} Q(x)\, dx + \int_{0}^{\infty} Q(x)\, dx$.

Thus, if $Q(z)$ satisfies the following two conditions:

- $Q(z)$ has only finitely many singularities, say $\alpha_1, \alpha_2, \ldots, \alpha_n$ in the upper half of the complex plane, and none on the real axis.
  Hence, when $R > |\alpha_j|$, for every $j = 1, \ldots, n$, residue theorem implies:
  $$\int_{C_R} Q(z)\, dz = 2\pi\mathbf{i} \sum_{j=1}^{n} \operatorname*{Res}_{z=\alpha_j} (Q(z)) \ .$$
- $\displaystyle\lim_{R\to\infty} \int_{\gamma_R} Q(z)\, dz = 0.$
  (This is going to involve the kind of argument we saw for example in HW 5, problem 12; Mid Term 2, problem 2. These arguments were based on finding a bound on the function, and using our important inequality. See examples §27.4, 27.5 below.)

Then, we will be able to conclude that
$$\text{P. V.} \int_{-\infty}^{\infty} Q(x)\, dx = 2\pi\mathbf{i} \sum_{j=1}^{n} \operatorname*{Res}_{z=\alpha_j} (Q(z))$$

(27.2) Example 1.– Let $0 < a < 1$. Compute $\displaystyle\int_0^{2\pi} \frac{1}{1 + a\sin(\theta)}\, d\theta$.

Solution. Set $z = e^{\mathbf{i}\theta}$, so that $\sin(\theta) = \dfrac{z - z^{-1}}{2\mathbf{i}}$ and $d\theta = \dfrac{1}{\mathbf{i}z} dz$. Our integral becomes:
$$\int_0^{2\pi} \frac{1}{1 + a\sin(\theta)}\, d\theta = \int_C \frac{1}{1 + a\left(\frac{z-z^{-1}}{2\mathbf{i}}\right)} \frac{1}{\mathbf{i}z}\ dz.$$
where, C is the circle of radius 1, centered at 0.

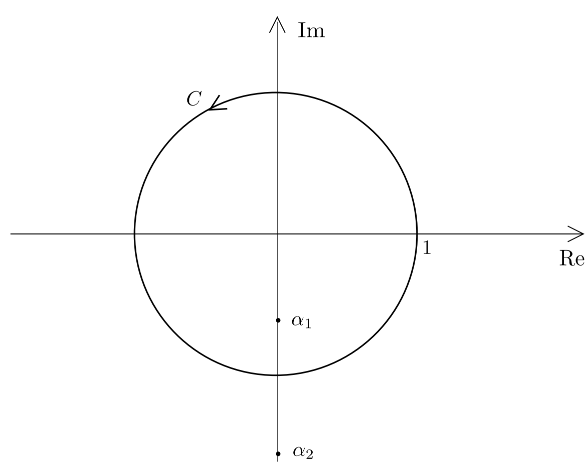

FIGURE 3. $\alpha_1 \in \text{Interior}(C)$ and $\alpha_2 \in \text{Exterior}(C)$.

We begin by simplifying the function we have to integrate.
$$\frac{1}{\mathbf{i}z\left(1 + a\left(\frac{z-z^{-1}}{2\mathbf{i}}\right)\right)} = \frac{2}{az^2 + 2\mathbf{i}z - a}.$$

Solving the quadratic equation $az^2 + 2\mathbf{i}z - a = 0$, we get two solutions α_1 and α_2: $az^2 + 2\mathbf{i}z - a = a(z-\alpha_1)(z-\alpha_2)$ where,

$$\alpha_1 = \mathbf{i}\frac{\sqrt{1-a^2}-1}{a}, \qquad \alpha_2 = \mathbf{i}\frac{-\sqrt{1-a^2}-1}{a}.$$

Since $0 < a < 1$, we get that $|\alpha_2| = \dfrac{1+\sqrt{1-a^2}}{a} > 1$. As $\alpha_1\alpha_2 = -1$ (set $z = 0$ in $az^2 - 2\mathbf{i}z - a = a(z-\alpha_1)(z-\alpha_2)$), $|\alpha_1| < 1$. So, of the two singularities, α_1 is within C and α_2 is outside of C (see Figure 3 above).

Now we can finish the computation, as follows.

$$\int_C \frac{2}{a(z-\alpha_1)(z-\alpha_2)}\,dz = 2\pi\mathbf{i}\frac{2}{a(\alpha_1-\alpha_2)} = \frac{4\pi\mathbf{i}}{a\cdot\frac{2\mathbf{i}\sqrt{1-a^2}}{a}} = \frac{2\pi}{\sqrt{1-a^2}}.$$

(27.3) Example 2.– Let $0 < p < 1$. Compute $\displaystyle\int_0^{2\pi} \frac{1}{1-2p\cos(\theta)+p^2}\,d\theta$.

Solution. Again, we substitute $z = e^{\mathbf{i}\theta}$, turning $\cos(\theta) = \dfrac{z+z^{-1}}{2}$ and $d\theta = \dfrac{1}{\mathbf{i}z}dz$. C is again the counterclockwise circle of radius 1, centered at 0. Thus,

$$\int_0^{2\pi} \frac{1}{1-2p\cos(\theta)+p^2}\,d\theta = \int_C \frac{1}{1-2p\left(\frac{z+z^{-1}}{2}\right)+p^2}\,\frac{1}{\mathbf{i}z}\,dz.$$

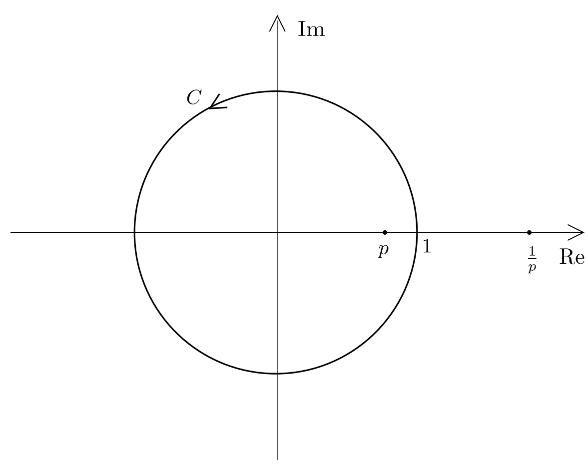

FIGURE 4. $p \in \text{Interior}(C)$ and $\dfrac{1}{p} \in \text{Exterior}(C)$.

Simplify the function to be integrated first:

$$\frac{1}{\mathbf{i}z\left(1-2p\left(\frac{z+z^{-1}}{2}\right)+p^2\right)} = \frac{1}{\mathbf{i}(z-pz^2-p+p^2z)} = \frac{1}{\mathbf{i}(1-pz)(z-p)}.$$

Thus the two singularities of this function are $z = p$ (within C) and $z = \frac{1}{p}$ (outside of C) - see Figure 4 above. Hence,

$$\int_C \frac{1}{\mathbf{i}(z-p)(1-pz)}\,dz = 2\pi\mathbf{i}\frac{1}{\mathbf{i}(1-p^2)} = \frac{2\pi}{1-p^2}.$$

(27.4) Example 3.– Compute $\displaystyle\int_{-\infty}^{\infty} \frac{1}{(x^2+1)^3}\, dx$.

Solution. Note that $\dfrac{1}{(x^2+1)^3}$ is an even function of x. So, there is no difference between P. V. $\int_{-\infty}^{\infty}$ and $\int_{-\infty}^{\infty}$.

Let $R > 1$ and consider the contour C_R as shown in Figure 5 below.

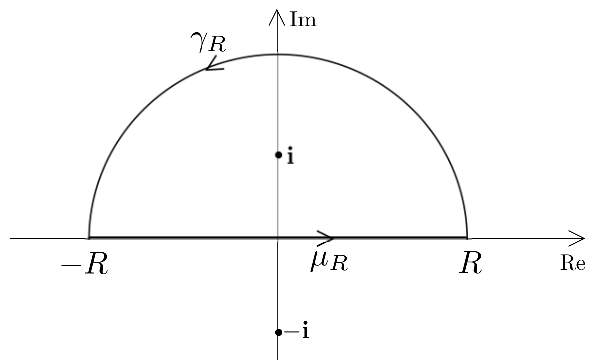

FIGURE 5. C_R consists of a straight line and a semicircle. With $R > 1$, $\mathbf{i} \in \text{Interior}(C)$ and $-\mathbf{i} \in \text{Exterior}(C)$.

The contour integral is now easy to compute, using Cauchy's integral formula:

$$\int_{C_R} \frac{dz}{(z^2+1)^3} = \int_{C_R} \frac{dz}{(z-\mathbf{i})^3(z+\mathbf{i})^3} = 2\pi\mathbf{i}\frac{1}{2!}\left[\frac{d^2}{dz^2}\left(\frac{1}{(z+\mathbf{i})^3}\right)\right]_{\text{Set } z=\mathbf{i}} = \frac{3\pi}{8}.$$

While, for z on γ_R, we have the following inequality: (see Lecture 2, §2.6, page 6).

$$|z^2+1| \geq |z|^2 - 1 = R^2 - 1 \Rightarrow \left|\frac{1}{(z^2+1)^3}\right| \leq \frac{1}{(R^2-1)^3}.$$

Therefore, using our important inequality (Lecture 12, §12.7):

$$\left|\int_{\gamma_R} \frac{1}{(z^2+1)^3}\, dz\right| \leq \frac{1}{(R^2-1)^3} \cdot \pi R \to 0 \text{ as } R \to \infty.$$

Hence $\displaystyle\lim_{R\to\infty} \int_{\gamma_R} \frac{1}{(z^2+1)^3}\, dz = 0$, and we get the answer:

$$\int_{-\infty}^{\infty} \frac{1}{(x^2+1)^3}\, dx = \lim_{R\to\infty}\left(\int_{C_R} \frac{1}{(z^2+1)^3}\, dz - \int_{\gamma_R} \frac{1}{(z^2+1)^3}\, dz\right) = \frac{3\pi}{8}.$$

(27.5) Example 4.– Compute $\displaystyle\int_{0}^{\infty} \frac{1}{x^6+1}\, dx$.

Solution. Again, the function is even, so we have

$$\int_0^\infty \frac{1}{x^6+1}\,dx = \frac{1}{2}\,\text{P. V.}\int_{-\infty}^\infty \frac{1}{x^6+1}\,dx.$$

Now there are six solutions to $z^6+1=0$, three above the real line and three below (see Lecture 2, page 4 - where we studied this for the first time).

$$\alpha_1 = e^{\mathbf{i}\frac{\pi}{6}}, \qquad \alpha_2 = e^{\mathbf{i}\frac{\pi}{2}} = \mathbf{i}, \qquad \alpha_3 = e^{\mathbf{i}\frac{5\pi}{6}} = -e^{-\mathbf{i}\frac{\pi}{6}}.$$
$$\beta_1 = e^{-\mathbf{i}\frac{\pi}{6}} = \overline{\alpha_1}, \qquad \beta_2 = e^{-\mathbf{i}\frac{\pi}{2}} = -\mathbf{i}, \qquad \beta_3 = e^{-\mathbf{i}\frac{5\pi}{6}} = \overline{\alpha_3}.$$

(see Figure 6 below).

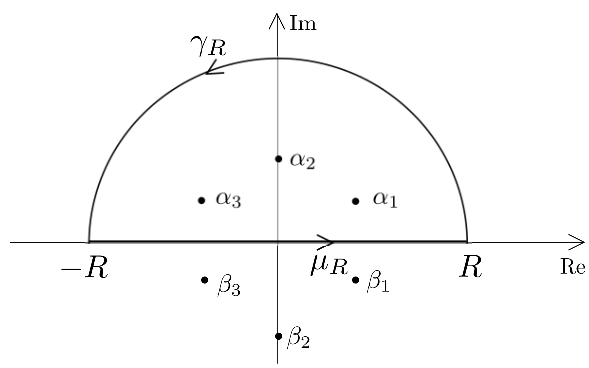

FIGURE 6. The contour C_R encloses $\alpha_1, \alpha_2, \alpha_3$. The other three solutions to $z^6=-1$ are in the lower half plane.

Again there are two steps to the problem. I am going to leave it to you to prove that $\lim_{R\to\infty}\int_{\gamma_R} \frac{1}{z^6+1}\,dz = 0$. The computation of the residues is a bit long, and given below.

$$\int_{C_R} \frac{1}{z^6+1}\,dz = 2\pi\mathbf{i}\left(\operatorname*{Res}_{z=\alpha_1}\left(\frac{1}{z^6+1}\right) + \operatorname*{Res}_{z=\alpha_2}\left(\frac{1}{z^6+1}\right) + \operatorname*{Res}_{z=\alpha_3}\left(\frac{1}{z^6+1}\right)\right).$$

Now each of these singularities is a pole of order 1. This allows us to compute the residue as follows:

$$\operatorname*{Res}_{z=\alpha_1}\left(\frac{1}{z^6+1}\right) = \lim_{z\to\alpha_1}\frac{z-\alpha_1}{z^6+1} = \lim_{z\to\alpha_1}\frac{1}{6z^5} = \frac{1}{6\alpha_1^5} \text{ (using l'hôpital rule).}$$

Using $\alpha_1^6=-1$, we get $\operatorname*{Res}_{z=\alpha_1}\left(\frac{1}{z^6+1}\right) = -\frac{\alpha_1}{6}$. Note that:

$$\alpha_1+\alpha_3 = e^{\mathbf{i}\frac{\pi}{6}} - e^{-\mathbf{i}\frac{\pi}{6}} = 2\mathbf{i}\sin(\pi/6) = \mathbf{i}.$$

$$\int_{C_R} \frac{1}{z^6+1}\,dz = -2\pi\mathbf{i}\frac{\alpha_1+\alpha_2+\alpha_3}{6} = -2\pi\mathbf{i}\frac{2\mathbf{i}}{6} = \frac{2\pi}{3}.$$

Hence $\displaystyle\int_0^\infty \frac{1}{x^6+1}\,dx = \frac{1}{2}\lim_{R\to\infty}\left(\int_{C_R}\frac{1}{z^6+1}\,dz - \int_{\gamma_R}\frac{1}{z^6+1}\,dz\right) = \frac{\pi}{3}.$