COMPLEX ANALYSIS: LECTURE 28

(28.0) Review..– Recall that in Lecture 27, we studied two types of real integrals which can be computed using residues.

Class I. $\displaystyle\int_0^{2\pi} R(\cos(\theta), \sin(theta))\, d\theta.$

Class II. $\displaystyle \text{P. V.} \int_{-\infty}^{\infty} Q(x)\, dx.$

In these notes we are going to take up two more types of examples.

Class III. Let $m \in \mathbb{R}_{>0}$. The following integrals are often computed using Jordan's lemma (see §28.1 and the example from §28.2 below).

$$\text{P. V.} \int_{-\infty}^{\infty} \cos(mx) Q(x)\, dx \qquad \text{and} \qquad \text{P. V.} \int_{-\infty}^{\infty} \sin(mx) Q(x)\, dx$$

The basic idea behind this type is still the one from Lecture 27, §27.4, §27.5. Here we merely notice that the two integrals written above are (respectively) the real and imaginary parts of $\displaystyle \text{P. V.} \int_{-\infty}^{\infty} e^{\mathbf{i}mx} Q(x)\, dx.$

We will integrate $e^{\mathbf{i}z}Q(z)$ over the contour C_R which consists of the straight line μ_R joining $-R$ to R, and the semicircle in the upper half plane γ_R, of radius R centered at 0 (see Figure 4 below). Jordan's lemma is need to conclude that $\displaystyle\lim_{R\to\infty} \int_{\gamma_R} e^{\mathbf{i}mz} Q(z)\, dz = 0.$

Class IV. In case $Q(z)$ has a pole (of order 1) on the real line, we will have to *indent* the contour, to avoid that point. See Figure 1 below, where 0 is avoided by going over it.

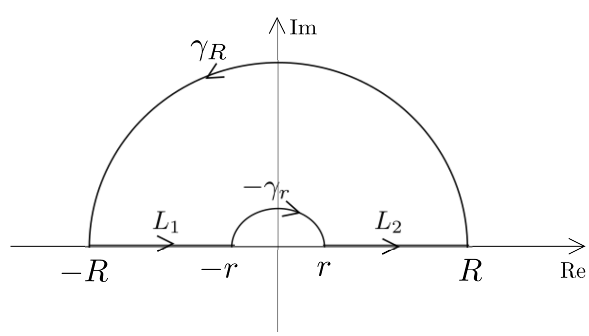

FIGURE 1. Contour $C_{r,R}$ consists of 4 smooth pieces: counterclockwise semicircle γ_R followed by straight line L_1, clockwise semicircle $-\gamma_r$ and another straight line L_2.

The outline for solving such problems is given in §28.3 below. To deal with the limit $r \to 0$, we are going to prove a technical lemma in §28.4. An example of such type is given in §28.5.

(28.1) Jordan's lemma.– [1]

Lemma. *Let $m \in \mathbb{R}_{>0}$. Let $Q(z)$ be a holomorphic function satisfying the following properties.*

- *There is a positive real number $R_0 \in \mathbb{R}_{>0}$ such that the set $\{z \in \mathbb{C} : \mathrm{Im}(z) \geq 0 \text{ and } |z| > R_0\}$ is in the domain of Q. See Figure 2 below.*

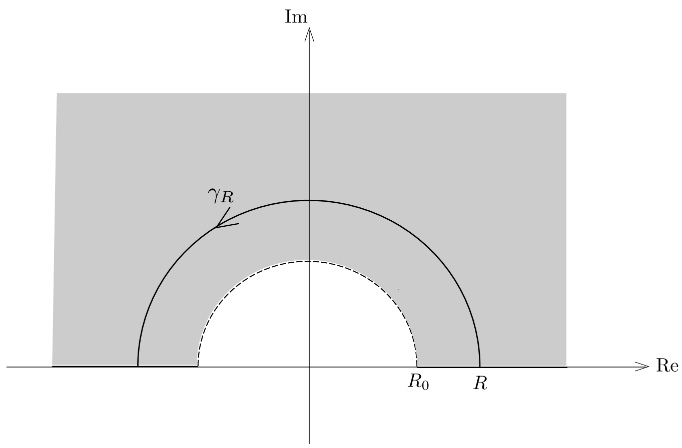

FIGURE 2. Domain of $Q(z)$ contains the set $|z| > R_0$ and $\mathrm{Im}(z) \geq 0$.

- *For every $R > R_0$, let γ_R be the semicircle in the upper half plane, of radius R, centered at 0. Then, there is a constant M_R (depending on R) such that $|Q(z)| < M_R$ for every z on the semicircle γ_R. Moreover, $\lim_{R\to\infty} M_R = 0$.*

Then,

$$\boxed{\lim_{R\to\infty} \int_{\gamma_R} e^{\mathbf{i}mz} Q(z)\, dz = 0}$$

Proof. The proof of this lemma is based on the following inequality.

Claim. For every $0 \leq \theta \leq \frac{\pi}{2}$, we have $\sin(\theta) \geq \frac{2\theta}{\pi}$.

Assuming this, we can finish the proof of the lemma as follows. Note that for $z = \gamma_R(\theta) = R(\cos(\theta) + \mathbf{i}\sin(\theta))$, we have: $|e^{\mathbf{i}mz}| = e^{-mR\sin(\theta)}$. Using this, we can bound the integral over γ_R, starting from the definition $\int_\gamma f(z)\, dz = \int_0^\pi f(\gamma(\theta))\gamma'(\theta)\, d\theta$:

[1]Camille Jordan (1838-1922) *Cours d'Analyse II (1894).*

$$\begin{aligned}
\left|\int_{\gamma_R} e^{\mathbf{i}mz} Q(z)\,dz\right| &= \left|\int_0^{\pi} e^{\mathbf{i}mR(\cos(\theta)+\mathbf{i}\sin(\theta))} Q(Re^{\mathbf{i}\theta})\mathbf{i}Re^{\mathbf{i}\theta} d\theta\right| \\
&< M_R R \int_0^{\pi} e^{-mR\sin(\theta)}\,d\theta \quad (\text{since } |Q(Re^{\mathbf{i}\theta})| < M_R) \\
&= 2M_R R \int_0^{\frac{\pi}{2}} e^{-mR\sin(\theta)}\,d\theta \quad (\text{since } \sin(\theta) = \sin(\pi-\theta)) \\
&\leq 2M_R R \int_0^{\frac{\pi}{2}} e^{-mR\frac{2\theta}{\pi}}\,d\theta \quad (\text{using the inequality written above}) \\
&= 2M_R R \left[-\pi \frac{e^{-2mR\frac{\theta}{\pi}}}{2mR}\right]_{\theta=0}^{\frac{\pi}{2}} = \frac{\pi M_R}{m}\left(1-e^{-mR}\right).
\end{aligned}$$

Now, by the assumption $\lim_{R\to\infty} M_R = 0$, we get that $\lim_{R\to\infty} \frac{\pi M_R}{m}\left(1-e^{-mR}\right) = \frac{\pi}{m}\lim_{R\to\infty} M_R = 0$. The lemma is proved, except for the claimed inequality.

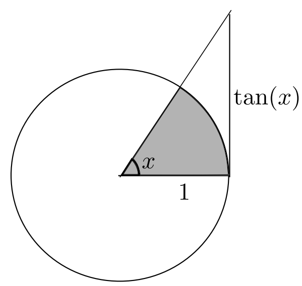

Figure 3. For $0 < x < \frac{\pi}{2}$, area of the shaded region $= \frac{x}{2} <$ area of the triangle $= \frac{\tan(x)}{2}$.

Proof of the claim. It is perhaps easy to draw the graph of $y = \sin(x)$ and $y = \frac{2x}{\pi}$, to see this inequality. Alternately, we can show that $f(x) = \frac{\sin(x)}{x}$ is a decreasing function of x, for $0 \leq x \leq \pi/2$, by showing that $f'(x) < 0$ for $0 < x < \pi/2$. This will imply that the smallest value $f(x)$ takes is at $x = \pi/2$, hence $\frac{\sin(x)}{x} \geq \frac{2}{\pi}$ for every $x \in [0, \pi/2]$.

Now, for $x \in (0, \pi/2)$, $f'(x) = \frac{x\cos(x) - \sin(x)}{x^2} = \frac{\cos(x)}{x^2}(x - \tan(x))$. As $\frac{\cos(x)}{x^2} > 0$, and $x - \tan(x) < 0$ (see Figure 3 above), we conclude that $f'(x) < 0$.

□

(28.2) Example 1.– Let us compute the integral $\int_0^{\infty} \frac{x\sin(x)}{x^2+a^2}\,dx$, where $a \in \mathbb{R}_{>0}$.

The idea is essentially same as the one from Lecture 27, §27.4 and §27.5. The only difference is that now we change $\sin(x)$ to the imaginary part of $e^{\mathbf{i}x}$.

Consider the following contour integral $\displaystyle\int_{C_R} \frac{ze^{\mathbf{i}z}}{z^2+a^2}\,dz$. The contour C_R is same as the one from Lecture 27, and is shown in Figure 4 below.

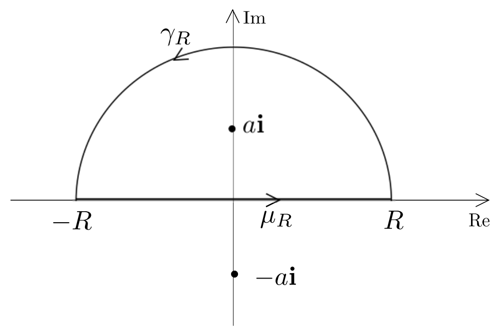

FIGURE 4. Contour C_R

1. By Cauchy's integral formula $\displaystyle\int_{C_R} \frac{ze^{\mathbf{i}z}}{(z+a\mathbf{i})(z-a\mathbf{i})}\,dz = 2\pi\mathbf{i}\frac{a\mathbf{i}e^{-a}}{2a\mathbf{i}} = \pi\mathbf{i}e^{-a}$.

2. We are going to apply Jordan's lemma from §28.1 above, to $Q(z) = \dfrac{z}{z^2+a^2}$ (with $m=1$). Clearly the set $\{z \in \mathbb{C} : |z| > a\}$ is in the domain of this function. On γ_R, this function is bounded by $\left|\dfrac{z}{z^2+a^2}\right| < \dfrac{R}{R^2-a^2} = M_R$, and $\displaystyle\lim_{R\to\infty} M_R = 0$.

So, Jordan's lemma applies, and we get $\displaystyle\lim_{R\to\infty}\int_{\gamma_R} e^{\mathbf{i}z}Q(z) = 0$.

3. Combining the previous two steps with the following observation (since the function is even):

$$\int_0^\infty \frac{x\sin(x)}{x^2+a^2}\,dx = \frac{1}{2}\,\text{P. V.}\int_{-\infty}^{\infty}\frac{x\sin(x)}{x^2+a^2}\,dx = \frac{1}{2}\,\text{Im}\left(\lim_{R\to\infty}\int_{\mu_R}\frac{ze^{\mathbf{i}z}}{z^2+a^2}\,dz\right),$$

we get:

$$\begin{aligned}\int_0^\infty \frac{x\sin(x)}{x^2+a^2}\,dx &= \frac{1}{2}\,\text{Im}\left(\lim_{R\to\infty}\left(\int_{C_R}\frac{ze^{\mathbf{i}z}}{z^2+a^2}\,dz - \int_{\gamma_R}\frac{ze^{\mathbf{i}z}}{z^2+a^2}\,dz\right)\right)\\ &= \frac{1}{2}\,\text{Im}(\pi\mathbf{i}e^{-a}) = \frac{\pi}{2}e^{-a}.\end{aligned}$$

(28.3) Indenting the contour.– An indented contour (for example, the one given in Figure 1 on page 1) consists of 4 smooth pieces. $C_{r,R} = \gamma_R + L_1 - \gamma_r + L_2$. It is used to compute the integrals of the Class II, III type: $\displaystyle\text{P. V.}\int_{-\infty}^{\infty} e^{\mathbf{i}mx}Q(x)\,dx$, where $m \in \mathbb{R}_{\geq 0}$.

The only difference is that the function $Q(z)$ is now allowed to have a pole of order 1 on the real line, assumed to be at $0 \in \mathbb{R}$ for simplicity. In this case, our steps are going to be:

- Compute $\displaystyle\int_{C_{r,R}} e^{\mathbf{i}mz}Q(z)\,dz$ using Cauchy's integral formula.
- Prove that $\displaystyle\lim_{R\to\infty}\int_{\gamma_R} e^{\mathbf{i}mz}Q(z)\,dz = 0$, either using our important inequality, or Jordan's lemma.
- Compute $\displaystyle\lim_{r\to 0}\int_{\gamma_r} e^{\mathbf{i}mz}Q(z)\,dz$ using the result given in §28.4 below.

Then, the final answer is:

$$\text{P. V.}\int_{-\infty}^{\infty} e^{\mathbf{i}mz}Q(z)\,dz = \lim_{\substack{R\to\infty\\ r\to 0}}\int_{L_1+L_2} e^{\mathbf{i}mz}Q(z)\,dz = \int_{C_{r,R}} e^{\mathbf{i}mz}Q(z)\,dz + \lim_{r\to 0}\int_{\gamma_r} e^{\mathbf{i}mz}Q(z)\,dz.$$

(28.4) .–

Lemma. *Let $f(z)$ be a holomorphic function. Assume that $x_0 \in \mathbb{R}$ is a pole of f, of order 1, with $\operatorname*{Res}_{z=x_0}(f(z)) = B \in \mathbb{C}$.*

Let $\gamma_r(\theta) = x_0 + re^{\mathbf{i}\theta}$, $0 \le \theta \le \pi$. Thus, γ_r is the counterclockwise semicircle of radius r, centered at x_0. Then,

$$\boxed{\lim_{r\to 0}\int_{\gamma_r} f(z)\,dz = \pi\mathbf{i}B}$$

(x_0 does not have to be on the real line for the validity of this lemma. It is just that we are only going to apply it in examples when x_0 is real.)

Proof. Since $f(z)$ has a pole of order 1 at $z = x_0$, its Laurent series expansion has the form (for some $R > 0$).

$$f(z) = \frac{B}{z - x_0} + \sum_{k\ge 0} a_k(z - x_0)^k \text{ for } z \in \mathbb{C} \text{ such that } 0 < |z - x_0| < R.$$

The function $g(z) = \displaystyle\sum_{k\ge 0} a_k z^k$ is then holomorphic on the disc $D(x_0; R)$. Pick $0 < \rho < R$ and let M be the absolute maximum of $|g(z)|$ on the closed disc $\overline{D(x_0;\rho)} = \{z : |z - x_0| \le \rho\}$.

For every $r < \rho$, we have: $\displaystyle\left|\int_{\gamma_r} g(z)\,dz\right| \le M\pi r \to 0$ as $r \to 0$.

For the term $\dfrac{B}{z - x_0}$, we can compute the integral directly from the definition:

$$\int_{\gamma_r} \frac{B}{z - x_0}\,dz = B\int_0^{\pi} \frac{1}{re^{\mathbf{i}\theta}} r\mathbf{i}e^{\mathbf{i}\theta}\,d\theta = \pi\mathbf{i}B.$$

Hence, $\lim_{r\to 0}\int_{\gamma_r} f(z)\,dz = \pi\mathbf{i}B + \lim_{r\to 0}\int_{\gamma_r} g(z)\,dz = \pi\mathbf{i}B.$ □

(28.5) Example 2.– Let $m > 0, a > 0$ be two real numbers. Compute $\displaystyle\int_0^\infty \frac{\sin(mx)}{x(x^2+a^2)}\,dx.$

Solution. The function is even. So, the integral can be written as

$$\frac{1}{2}\,\mathrm{Im}\left(\mathrm{P.V.}\int_{-\infty}^{\infty}\frac{e^{\mathbf{i}mx}}{x(x^2+a^2)}\,dx\right) = \frac{1}{2}\,\mathrm{Im}\left(\lim_{\substack{R\to\infty\\ r\to 0}}\int_{L_1+L_2}\frac{e^{\mathbf{i}mz}}{z^2+a^2}\,dz\right) \text{ (see Figure 5 below).}$$

We will compute it using the steps outlined in §28.3 above. Let $C_{r,R}$ be the contour as shown below.

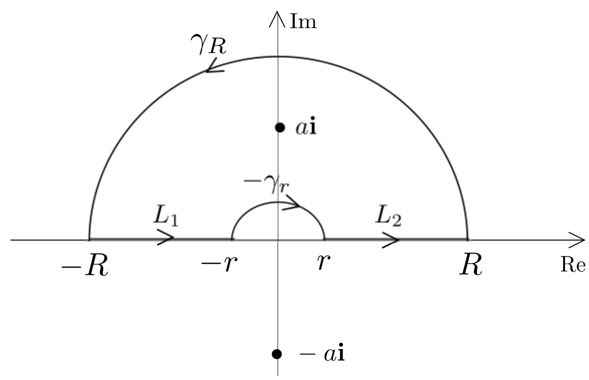

FIGURE 5. Contour $C_{r,R}$ indented at 0.

1. Using Cauchy's integral formula: $\displaystyle\int_{C_{r,R}}\frac{e^{\mathbf{i}mz}}{z(z^2+a^2)}\,dz = 2\pi\mathbf{i}\frac{e^{-ma}}{a\mathbf{i}(2a\mathbf{i})} = -\pi\mathbf{i}\frac{e^{-ma}}{a^2}.$

2. Verify that Jordan's lemma (§28.1) applies to $Q(z) = \dfrac{1}{z(z^2+a^2)}$ (left as an easy exercise), so that $\displaystyle\lim_{R\to\infty}\int_{\gamma_R} e^{\mathbf{i}mz}Q(z)\,dz = 0.$

3. The function $f(z) = \dfrac{e^{\mathbf{i}mz}}{z(z^2+a^2)}$ has a pole of order 1 at $z = 0$, with $\operatorname*{Res}_{z=0}(f(z)) = \lim_{z\to 0} zf(z) = \dfrac{1}{a^2}$. So, by Lemma 28.4 above, we get: $\displaystyle\lim_{r\to 0}\int_{\gamma_r} f(z)\,dz = \mathbf{i}\frac{\pi}{a^2}.$

Combining all this, we get: $\displaystyle\mathrm{P.V.}\int_{-\infty}^{\infty}\frac{e^{\mathbf{i}mx}}{x(x^2+a^2)}\,dx = \mathbf{i}\frac{\pi}{a^2}(1-e^{-ma})$. Hence, our answer is:

$$\int_0^\infty \frac{\sin(mx)}{x(x^2+a^2)}\,dx = \frac{\pi}{2a^2}\left(1-e^{-ma}\right).$$