COMPLEX ANALYSIS: LECTURE 29

(29.0) What is in these notes.– These notes contain some remarkable applications of the results we have obtained so far [1]. Namely,

(1) Cauchy's residue theorem (Lecture 26, §26.4).

(2) Weierstrass' theorem on uniform convergence (Lecture 22, §22.6).

(3) Our important inequality (Lecture 12, §12.7).

We are going to use these to prove various expressions involving trigonometric functions, almost all of them due to Euler. For instance:

$$\cot(z) = \frac{1}{z} + \sum_{n=1}^{\infty}\left(\frac{1}{z-n\pi} + \frac{1}{z+n\pi}\right).$$

$$\operatorname{cosec}(z) = \frac{1}{z} + \sum_{n=1}^{\infty}(-1)^n\left(\frac{1}{z-n\pi} + \frac{1}{z+n\pi}\right).$$

$$\frac{\sin(z)}{z} = \prod_{n=1}^{\infty}\left(1 - \frac{z^2}{n^2\pi^2}\right).$$

These identities can be viewed as approximating trigonometric functions by rational functions. The first two can also be thought of as "partial fraction decomposition", except now there are infinitely many terms.

While these and many other identities were obtained by Euler for computational purposes, he used the last one to obtain the explicit value of $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^2}$. (You can read more about it on wikipedia - look for *Basel problem.*)

Euler's solution to the Basel problem. Assuming the validity of $\displaystyle\frac{\sin(z)}{z} = \prod_{n=1}^{\infty}\left(1-\frac{z^2}{n^2\pi^2}\right)$, we are going to compare the coefficient of z^2 on both sides.

$$\frac{\sin(z)}{z} = 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \cdots \quad \text{from the Taylor series expansion of } \sin(z).$$

[1] This material is optional. We are going to use the infinite product expansion of $\frac{\sin(z)}{z}$ in connection with Euler's gamma function later (Lecture 31, 32). You will have to take it on faith, in case you choose not to read these notes.

So, the left–hand side gives us $-\dfrac{1}{6}$. The right–hand side gives us an infinite sum, namely $-\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^2\pi^2}$. Hence:

$$-\frac{1}{6} = -\sum_{n=1}^{\infty}\frac{1}{n^2\pi^2} \Rightarrow \boxed{\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}}$$

I encourage you to read more about Euler's work on infinite series in:

V.S. Varadarajan. *Euler and his work on infinite series.* Bulletin of the American Mathematical Society (2007) volume 44, no. 4, 515-539.

You can download this paper from the following link:

`https://people.math.osu.edu/gautam.42/S20/Varadarajan-Euler.pdf`

(29.1) Generalized partial fractions.– Certain type of meromorphic functions can be written as an infinite sum of rational fractions. This result is due to Mittag-Lefler [2] who proposed and solved the problem of contructing meromorphic functions from the knowledge of their behaviour near poles (known as Mittag-Leffler theorem). The result given in this section is based on his paper from 1880, and is a special case of Mittag-Leffler theorem.

Let $f : \mathbb{C} \dashrightarrow \mathbb{C}$ be a meromorphic function (see Lecture 26, §26.5, §26.6). Let $A \subset \mathbb{C}$ be the set of poles of f in the complex plane. Let us assume that the poles are arranged, $A = \{a_1, a_2, \ldots\}$ in a way that $0 < |a_1| \leq |a_2| \leq \cdots$ I am assuming that 0 is not a pole of f[3].

Assumption 1. Assume that all the poles of f are of order 1 (poles of order 1 are also called *simple poles*), and let $\operatorname*{Res}_{z=a_k}(f(z)) = b_k$ for every $k \geq 1$.

Assumption 2. For every $m \in \mathbb{Z}_{\geq 1}$, it is possible to pick $R_m \in \mathbb{R}_{>0}$, so that

(1) $R_1 < R_2 < \ldots$ and $R_m \to \infty$ as $m \to \infty$.
(2) $|a_j| \neq R_m$ (for every $j, m \geq 1$). (Meaning, the poles of f do not lie on the circle C_m of radius R_m, centered at 0).
(3) There is a constant $M \in \mathbb{R}_{>0}$ such that $|f(z)| < M$ for every z lying on C_m (for every $m \geq 1$). Just to clarify, M is independent of $m \geq 1$. The same constant is supposed to work for all circles.

[2]Gösta Mittag-Leffler (1846-1927)

[3]This is not a serious assumption for the result, since we can just replace $f(z)$ by $f(z) - \dfrac{b}{z}$, if $f(z)$ did have a pole of order 1 at $z = 0$ with residue b. See Example §29.2 below

Then, for every $w \in \mathbb{C} \setminus A$:

$$\boxed{f(w) = f(0) + \sum_{k=1}^{\infty} b_k \left(\frac{1}{w - a_k} + \frac{1}{a_k} \right)}$$

Proof. Let $\Omega = \mathbb{C} \setminus A$, so that $f : \Omega \to \mathbb{C}$ is holomorphic. For each $m \in \mathbb{Z}_{\geq 1}$, let C_m be the counterclockwise oriented circle of radius R_m, centered at 0.

Let $N_m \in \mathbb{Z}_{\geq 1}$ denote the positive integer so that

$$a_1, a_2, \ldots, a_{N_m} \in \text{Interior}(C_m) \qquad \text{and} \qquad a_n \in \text{Exterior}(C), \text{ for every } n > N_m.$$

The way the poles are arranged gives us $N_1 \leq N_2 \leq \ldots$

Consider a point $w \in \Omega$. Using Cauchy's integral formula, we have (recall that each a_k is a pole of order 1, with residue b_k):

$$\frac{1}{2\pi \mathbf{i}} \int_{C_m} \frac{f(z)}{z - w}\, dz = f(w) + \sum_{k=1}^{N_m} \frac{b_k}{b_k - w} \ .$$

Replacing $\dfrac{1}{z - w} = \dfrac{1}{z}\left(1 + \dfrac{w}{z - w}\right)$, we get (again by Cauchy's integral formula):

$$\begin{aligned} \frac{1}{2\pi \mathbf{i}} \int_{C_m} \frac{f(z)}{z - w}\, dz &= \frac{1}{2\pi \mathbf{i}} \int_{C_m} \frac{f(z)}{z}\, dz + \frac{w}{2\pi \mathbf{i}} \int_{C_m} \frac{f(z)}{z(z - w)}\, dz \\ &= f(0) + \sum_{k=1}^{N_m} \frac{b_k}{a_k} + \frac{w}{2\pi \mathbf{i}} \int_{C_m} \frac{f(z)}{z(z - w)}\, dz \ . \end{aligned}$$

Combining the two expressions, we get:

$$f(w) - \left(f(0) + \sum_{k=1}^{N_m} b_k \left(\frac{1}{w - a_k} + \frac{1}{a_k} \right) \right) = \frac{w}{2\pi \mathbf{i}} \int_{C_m} \frac{f(z)}{z(z - w)}\, dz.$$

Assume that m is large enough so that $|w| < R_m$. Then we get, using $|f(z)| < M$ for every z on C_m, and $|z - w| \geq R_m - |w|$:

$$\left| \frac{w}{2\pi \mathbf{i}} \int_{C_m} \frac{f(z)}{z(z - w)}\, dz \right| < \frac{|w|}{2\pi} \frac{M}{R_m(R_m - |w|)} 2\pi R_m = \frac{M|w|}{R_m - |w|}.$$

By our assumption $R_m \to \infty$ as $m \to \infty$. So this integral goes to 0, as $m \to \infty$. Hence we obtain:

$$f(w) = f(0) + \sum_{k=1}^{\infty} b_k \left(\frac{1}{w - a_k} + \frac{1}{a_k} \right).$$

Note that the same argument also gives the uniform convergence of this infinite sum, as follows. Let $K \subset \mathbb{C}$, a compact set be given such that $K \subset \Omega$. Additionally, let $\varepsilon > 0$ be given. Let $a \in \mathbb{R}_{>0}$ be such that $|w| < a$ for every $w \in K$ (exists since K is bounded).

Choose m large enough that $\dfrac{Ma}{R_m - a} < \varepsilon$. Take $N = N_m$. Then, for every $w \in K$, we get:

$$\left|\sum_{k=N+1}^{\infty} b_k \left(\frac{1}{w-a_k} + \frac{1}{a_k}\right)\right| = \left|\frac{w}{2\pi\mathbf{i}}\int_{C_m} \frac{f(z)}{z(z-w)}\, dz\right| < \frac{Ma}{R_m - a} < \varepsilon.$$

□

(29.2) Example.– Let us derive the expression of $\operatorname{cosec}(z)$ as given in §29.0. The poles of $\operatorname{cosec}(z)$ are at $\pi\mathbb{Z}$. By l'hôpital rule ($n \in \mathbb{Z}$):

$$\lim_{z\to n\pi}(z-n\pi)\operatorname{cosec}(z) = \lim_{z\to n\pi}\frac{z-n\pi}{\sin(z)} = \lim_{z\to n\pi}\frac{1}{\cos(z)} = \frac{1}{\cos(n\pi)} = (-1)^n.$$

So, all the poles are simple, and $\operatorname*{Res}_{z=n\pi}(\operatorname{cosec}(z)) = (-1)^n$.

Consider the function $f(z) = \operatorname{cosec}(z) - \dfrac{1}{z}$, so that it does not have a pole at 0. It remains to check that Assumption 2 of §29.1 holds. Once that is done, the result obtained in §29.1 will give us:

$$\boxed{\operatorname{cosec}(z) = \frac{1}{z} + \sum_{n\in\mathbb{Z}_{\neq 0}} (-1)^n \left(\frac{1}{z-n\pi} + \frac{1}{n\pi}\right)}$$

(Compare with the expansion given in §29.0 above, where the terms with $n \in \mathbb{Z}_{\geq 1}$ and $-n$ are combined.)

Verification of Assumption 2. For each $m \in \mathbb{Z}_{\geq 1}$, let $R_m = \left(m + \dfrac{1}{2}\right)\pi$. Then,

- $R_1 < R_2 < \dots$ and $R_m \to \infty$ as $m \to \infty$.
- $\operatorname{cosec}(z) - \dfrac{1}{z}$ does not have poles on the circle C_m of radius R_m, centered at 0.

To find the uniform bound M, it would be sufficient to consider $|\operatorname{cosec}(z)|$ alone, since $\left|\operatorname{cosec}(z) - \dfrac{1}{z}\right| \leq |\operatorname{cosec}(z)| + \dfrac{1}{|z|} < |\operatorname{cosec}(z)| + 1$ (for z lying on any of C_m, $|z| = R_m > 1$).

Let us break the circle C_m into two parts (these are not disjoint, only $C_m = P_1 \cup P_2$ is needed).

$$P_1 = \left\{z \in C_m : |\operatorname{Im}(z)| \geq \frac{\pi}{8}\right\} \qquad \text{and} \qquad P_2 = \left\{z \in C_m : \left|z \pm \left(m + \frac{1}{2}\right)\pi\right| \leq \frac{\pi}{4}\right\}$$

See Figure 1 below.

For $z \in \mathbb{C}$ such that $|\operatorname{Im}(z)| \geq \dfrac{\pi}{8}$, we have

$$|\operatorname{cosec}(z)| = \frac{2}{|e^{\mathbf{i}z} - e^{-\mathbf{i}z}|} \leq \frac{2}{\big||e^{\mathbf{i}z}| - |e^{-\mathbf{i}z}|\big|} \leq \frac{2}{e^{|\operatorname{Im}(z)|} - e^{-|\operatorname{Im}(z)|}} < \frac{2}{e^{\pi/8} - 1}.$$

This gives us a bound (independent of m) on the part P_1 of C_m.

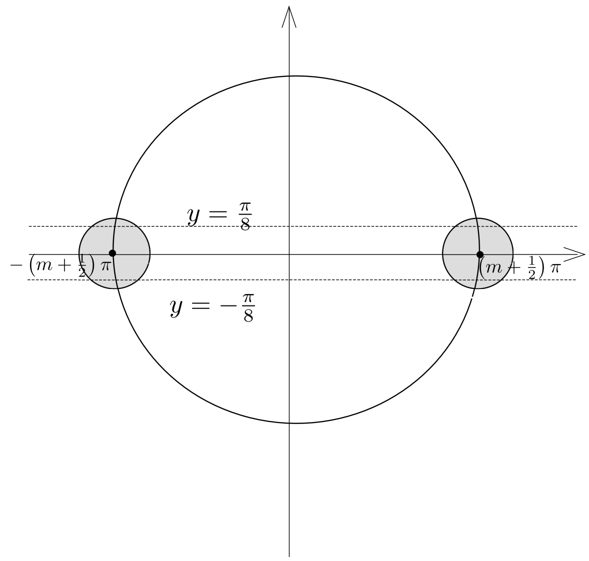

FIGURE 1. P_1 consists of part of C_m lying above $y = \dfrac{\pi}{8}$ or lying below $y = -\frac{\pi}{8}$. P_2 consists of part of C_m lying in the shaded discs, both of radii $\dfrac{\pi}{4}$.

For the part P_2, note that $|\operatorname{cosec}(z)| = |\operatorname{cosec}(z + k\pi)|$ for any $k \in \mathbb{Z}$. So, if M_1 is the absolute maximum of $|\operatorname{cosec}(z)|$ on the closed disc $\left\{z \in \mathbb{C} : \left|z - \frac{\pi}{2}\right| \leq \frac{\pi}{4}\right\}$, then (again independent of m), we have $|\operatorname{cosec}(z)| \leq M_1$ for every z lying in the part P_2 of C_m.

Now, take $M > \operatorname{Max}(M_1, 2/(e^{\pi/8} - 1))$.

(29.3) Weierstrass' infinite product expansion.– A result parallel to that of §29.1 was obtained by Weierstrass in 1876. Here we focus on zeroes of a holomorphic function, instead of poles of a meromorphic function.

Let $f : \mathbb{C} \to \mathbb{C}$ be an entire holomorphic function. Assume that f vanishes at (and only at) points of a set $A \subset \mathbb{C}$, which (as in §29.1 above) we arrange $A = \{a_1, a_2, \ldots\}$ so as to have $0 < |a_1| \leq |a_2| \leq \ldots$ Again, we are assuming that $f(0) \neq 0$, but it is not a serious assumption. (see Footnote 3 on page 2).

Assumption 1. The order of vanishing of f at each a_k is 1.

Assumption 2. Same as Assumption 2 of §29.1, but for the function $\dfrac{f'(z)}{f(z)}$.

Then, for every $z \in \mathbb{C}$:

$$\boxed{f(z) = f(0)e^{\frac{f'(0)}{f(0)}z} \prod_{n=1}^{\infty} \left(\left(1 - \frac{z}{a_n}\right) e^{\frac{z}{a_n}} \right)}$$

Remark. The notation $\prod_{k=1}^{\infty} u_k(z)$ if for *infinite product*. Explicitly, we consider the following sequence of functions:

$$F_1(z) = u_1(z), \quad F_2(z) = u_1(z) \cdot u_2(z), \quad \cdots \quad F_n(z) = u_1(z) \cdot u_2(z) \cdots u_n(z), \quad \cdots$$

Then, $\prod_{k=1}^{\infty} u_k(z)$ is the limit $\lim_{n\to\infty} F_n(z)$. In order to ensure that the limit behaves well, we will have to verify that the sequence of functions $\{F_n(z)\}_{n=1}^{\infty}$ converges uniformly (Weierstrass' theorem on uniform convergence §22.6).

Proof. The proof in fact follows from the result obtained in §29.1, applied to the function $g(z) = \frac{f'(z)}{f(z)}$. Note that $g(z)$ has simple poles at the points of A, of residue 1 (see Problem Set 7, problem 5: $f(a) = 0$ implies $\operatorname{Res}_{z=a}\left(\frac{f'(z)}{f(z)}\right)$ = order of vanishing of f at a).

Thus we get:

$$\frac{f'(z)}{f(z)} = \frac{f'(0)}{f(0)} + \sum_{n=1}^{\infty}\left(\frac{1}{z-a_n} + \frac{1}{a_n}\right)$$

We view this as a differential equation for $f(z)$, and try to write $f(z)$ as its solution.

The following calculation is heuristic, but very instructive. Note that $\frac{f'(z)}{f(z)} = \frac{d}{dz}(\log(f(z)))$ (it is therefore called *logarithmic derivative*). So, taking an antiderivative on both sides of the differential equation, we get (at least formally - i.e, without worrying about logarithms):

$$\log(f(z)) = C + \frac{f'(0)}{f(0)}z + \sum_{n=1}^{\infty}\left(\log(z-a_n) + \frac{z}{a_n}\right) ,$$

where C is a constant. Set $z = 0$ to get (again, we are not worrying about the convergence issues):

$$\log(f(0)) = C + \sum_{n=1}^{\infty}\log(-a_n).$$

Substituting it back in $\log(f(z))$ written above, we get:

$$\begin{aligned}\log(f(z)) &= \log(f(0)) + \frac{f'(0)}{f(0)}z + \sum_{n=1}^{\infty}\left(\log(z-a_n) - \log(-a_n) + \frac{z}{a_n}\right) \\ &= \log(f(0)) + \frac{f'(0)}{f(0)}z + \sum_{n=1}^{\infty}\left(\log\left(1-\frac{z}{a_n}\right) + \frac{z}{a_n}\right).\end{aligned}$$

Now take exponential to get the

$$f(z) = f(0)e^{\frac{f'(0)}{f(0)}z}\prod_{n=1}^{\infty}\left(\left(1-\frac{z}{a_n}\right)e^{\frac{z}{a_n}}\right).$$

Now that the final result is obtained, we can proceed as follows (not relying on the heuristic calculation). (1) Prove that the infinite product given on the right–hand side of the equation written above, converges uniformly on $\mathbb{C}$ [4]. (2) (easy) Check that it solves the same differential equation as $f(z)$, with the same initial value at $z = 0$. Hence, it equals $f(z)$. $\square$

(29.4) Example.– Consider the function $f(z) = \frac{\sin(z)}{z}$. It is an entire holomorphic function, with zeroes of order 1 at $z = n\pi$, $n \in \mathbb{Z}_{\neq 0}$. We have $f(0) = 1$ (by l'hôpital) and $\frac{f'(z)}{f(z)} = \cot(z) - \frac{1}{z}$, whose value at $z = 0$ is again computed using l'hôpital:

$$\lim_{z\to 0} \frac{z\cos(z) - \sin(z)}{z\sin(z)} = \lim_{z\to 0} \frac{-z\sin(z)}{z\cos(z) + \sin(z)} = \lim_{z\to 0} \frac{-\sin(z) - z\cos(z)}{2\cos(z) - z\sin(z)} = 0.$$

Leaving aside the verification of the assumptions imposed in §29.3, the result is:

$$\boxed{\frac{\sin(z)}{z} = \prod_{n\in\mathbb{Z}_{\neq 0}} \left(\left(1 - \frac{z}{n\pi}\right) e^{\frac{z}{n\pi}} \right)}$$

(Compare with the infinite product expansion written in §29.0 above, where, again we have grouped the terms n and $-n$ together).

Verification of Assumption 2 from §29.3 is absolutely similar to the one for $\mathrm{cosec}(z)$ given in §29.2, except this time we are going to have to find a bound on $\frac{f'(z)}{f(z)} = \cot(z) - \frac{1}{z}$. The argument given there works verbatim and I will not repeat it here.

[4] I am not going to write a proof here. In Lecture 32, we will discuss the uniform convergence of an infinite product defining the gamma function. A proof will be given then.